# Harmonisation en Mathématiques

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September 6, 2023

The following notes have been written for the course “Harmonisation en Mathématiques” taught at Université du Littoral during the academic year 2023-2024. Please direct any comments or questions to augustin.cosse@univ-littoral.fr.

## 1 Trigonometry

We consider an orthonormal frame $(0, \vec{u}, \vec{v})$ of the Euclidean space. We call trigonometric circle the circle of center $(0, 0)$ and of radius $r = 1$ (see Fig 2). For any point $P$ on the trigonometric circle, we use $\theta$ to denote the $(\vec{u}, OP)$ (in radians). Note that one radian is defined as the angle corresponding to an arc whose length is equal to the circle's radius. Stated otherwise, the measure (in radians) of any angle can be obtained by dividing the arc length by the radius of the circle.

### 1.1 Brief history

Although the idea of measuring angles by the length of the arc had already been used for a while (e.g. the Persian Mathematician Al-Kashi had already been using “diameter parts” as units), the notion of radians is usually credited to Roger Cotes (an English Mathematician from the 17th century, professor at Cambridge from 1707 until his death in 1716). The cousing of Cotes, Robert Smith collected and published Cotes mathematical writings in a book *Harmonia Mensurarum* (the Harmony of measures), and citing a note that Cotes that did not survive, provided the first definition of the radian.

> *“Now this number is equal to 180 degrees as the radius of a circle to the semicirconference, this is as 1 to 3.141592 ...”* (Cotes)

The number which is now known as $\pi$ first appeared in the work of the welsh mathematician William Jones in 1706. But the first rigorous algorithm to calculate the circumference of the circle was devised around 250 BC by the Greek mathematician Archimedes. Archimedes computed upper and lower bounds for $\pi$ by drawing a regular hexagon inside and outside the circle and doubled the number of sides until he reached a 96 sided regular polygon (see Fig 3). By calculating the perimeters of the

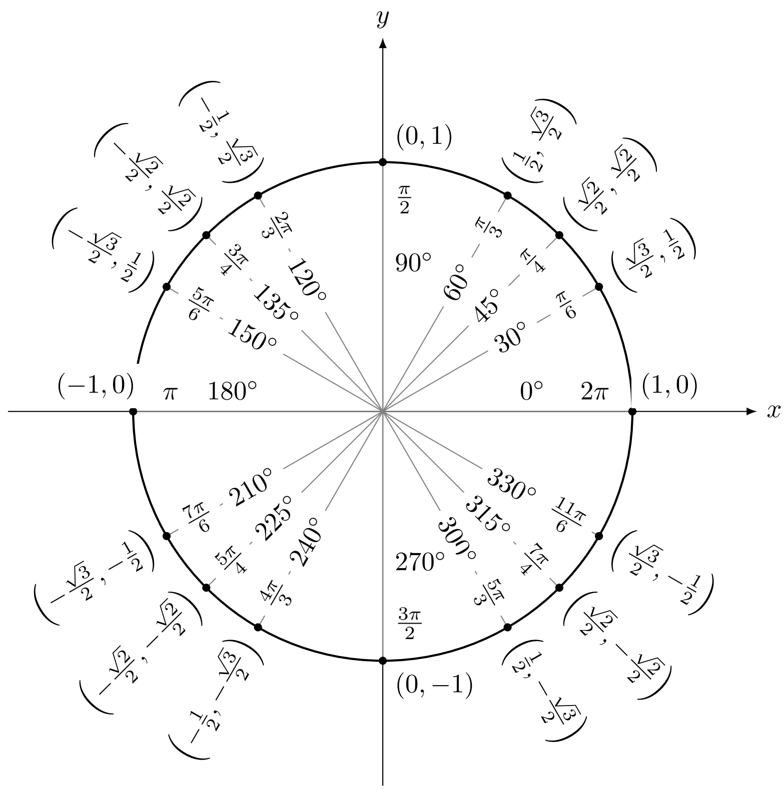



Figure 1: Commonly found trigonometric values

resulting two polygons, he was able to come up with the estimation

$$\frac{223}{71} < \pi < \frac{22}{7} \tag{1}$$

or $3.9408 < \pi < 3/1429$.

## 1.2 The trigonometric circle

Given the position of a point $\boldsymbol{P}$ and the corresponding angle on the trigonometric circle, one can show that the $x$ and $y$ axis coordinates of the point are respectively given by $\cos\theta$ and $\sin\theta$. In fact we have

$$\cos\theta = x/1 = \frac{\vec{OP}_x}{|OP|} \tag{2}$$

$$\sin\theta = y/1 = \frac{\vec{OP}_y}{|OP|} \tag{3}$$

The angle is determined up to $2\pi$. I.e. on the trigonometric circle, $\pi/2 = 5\pi/2 = \pi/2 + 2k\pi$ for any $k$. Commonly found trigonometric values are given in Fig 1 below

The following formulas can be proved using the Pythagorean theorem

$$\cos^2\theta + \sin^2\theta = 1 \tag{4}$$

$$\cos(-\theta) = \cos\theta \tag{5}$$

$$\sin(-\theta) = -\sin\theta \tag{6}$$

Similarly, one can prove the following formulas geometrically

$$\sin(a+b) = \sin a \cos b + \sin b \cos a \tag{7}$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{8}$$

**Exercise 1.** *Using the trigonometric circle, show formula* (7) *and* (8)*.*

**Exercise 2.** *For any two real numbers $a, b$, express* $\sin(a-b), \cos(a-b), \sin(2a), \cos(2a)$ *as functions of* $\cos a$*,* $\cos b$ $\sin a$ *and* $\sin b$

**Exercise 3.** *Show that* $\cos(\pi - a) = -\cos a$ *and* $\sin(\pi - a) = \sin a$*. Then show* $\sin(\pi/2 - a) = \cos a$*.*

**Exercise 4.** *Without the help of any calculator, find the exact value of* $\cos \pi/12$*.*

**Exercise 5.** *Show that* $\frac{\sin 2x}{\sin^2 x} = 2 \cot x$*.*

**Exercise 6.** *Find the exact value of* $\sin\left(\frac{5\pi}{6}\right)$

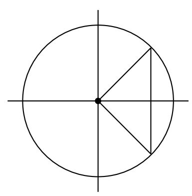
**Exercise 7.** *Express* $\sin^2$ *in terms of* $\cos(2x)$

**Exercise 8.** *Show that* $\sin(2x) \tan x = 1 - \cos(2x)$

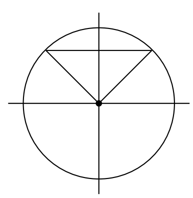
**Exercise 9.** *Show that*

$$\frac{\cos(2x)\sin(2x)}{2\cos^2 x} = \sin(2x) - \tan x$$

## 1.3 Solving simple trigonometric equations



- $\cos a = \cos b$ $\qquad x = a + 2k\pi, -a + 2k\pi,\ k \in \mathbb{Z}$



- $\sin a = \sin b$ $\qquad x = a + 2k\pi, \pi - a + 2k\pi,\ k \in \mathbb{Z}$

**Exercise 10.** *Solve the following equations*

$$\cos x - 5\sin(2x) = 0 \tag{9}$$
$$2\sin^2 x + 5\sin x + 3 = 0 \tag{10}$$
$$2\sin^2 x - \sin x = 0 \tag{11}$$
$$2\sin(2x) + \sqrt{2} = 0 \tag{12}$$

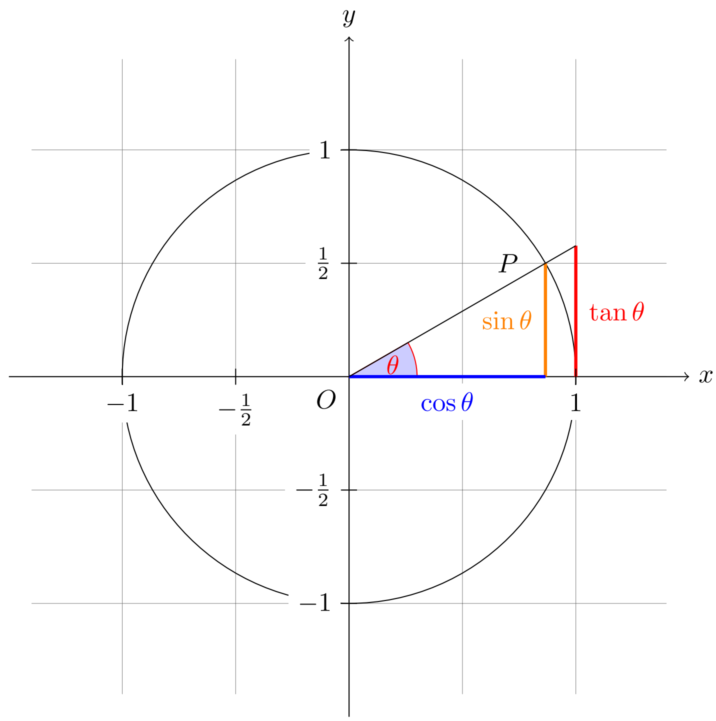



Figure 2: The trigonometric circle

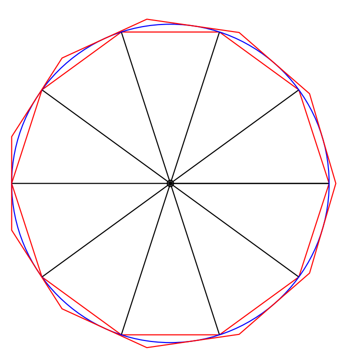


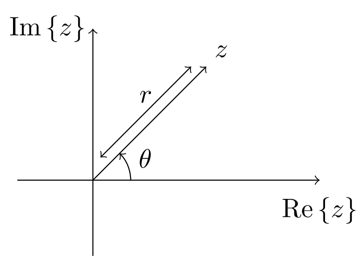
Figure 3: Archimedes polygons

## 1.4 Imaginary numbers

A number of the form $x + iy$ where $i^2 = -1$ (or $i = \sqrt{-1}$), and $x, y$ are real numbers, is called a *complex number*. The set of complex numbers is isomorphic (i.e. there exists a one-to-one correspondence between the elements of the two sets that can be encoded by a linear map preserving scalar multiplication and addition) to the $\mathbb{R}^2$ plane. With the operators of addition and multiplication, we denote this set as $\mathbb{C}$.

One of the first mathematicians to imagine the existence is Girolamo Cardano (Jérôme Cardan) who noticed that the equation $x(10 - x) = 40$ can be solved with the two solutions $5 \pm \sqrt{-15}$. Cardano indicates that the product of those numbers (although none of them does exist) gives the value 40. Cardano suggest the reader to be creative and describes those numbers as *"as subtle as they are useless"*. The first to formalize complex numbers is Raphaël Bombelli who is also the first to have set the rules to manipulate the numbers. The term *imaginary* was coined by *Descartes* (1637). Up to those times, mathematicians did not really know how to solve cubic equations in a systematic manner. The only approach they knew was through the use of so-called *depressed cubics* or equations of the form $x^3 + bx + c = 0$. The trick then consisted in rewriting general cubic equations as depressed cubics and then apply to the technique to the depressed equation. The problem with this approach is its limited applicability due to the presence of square roots. Solving depressed cubic equations requires solving a quadratic equation. This is precisely in this setting that Raphael Bombelli developed the first theory of imaginary numbers.

For any imaginary number $z = x + iy$ we call $x$ and $y$ the real and imaginary parts of $z$, $\text{Re}\{z\} = x$, $\text{Im}\{z\} = y$. Any imaginary or complex number can be represented as the pair $(x, y)$ in the $\mathbb{R}^2$ plane as the vector $\vec{z} = (x, y)$




We call the length of the vector connecting the origin to the point $(x, y)$ the *modulus* of $\vec{z}$ abd represent this quantity with the letter $r$. I.e.

$$r = |z| = \sqrt{x^2 + y^2} \tag{13}$$

Similarly, we call *argument* of the number $z$ the measure of the angle between the real axis and the vector $(x, y)$ and represent this quantity with the letter $\theta$

$$\theta = \arg\{z\} = \angle(0,0), (0,1), z \tag{14}$$

In the setting of imaginary numbers, the $x$-axis is known as the *real axis* and the $y$-axis is known as the *imaginary axis*. Any complex number $z$ with no real part will be located on the imaginary axis. Similarly, any complex number with no imaginary part ($y = 0$) will be located on the real axis.

For any complex number $z$, one can write the vector $\vec{z}$ as

$$\vec{z} = r\cos\theta \boldsymbol{e}_x + r\sin\theta \boldsymbol{e}_y \tag{15}$$

the addition and subtraction of complex numbers can be carried out by simply adding and subtracting the real and imaginary parts. I.e. if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$. Similarly, multiplication by a scalar corresponds to scaling the vector. I.e.

$$\lambda\vec{z} = \lambda x\boldsymbol{e}_x + \lambda y\boldsymbol{e}_y \tag{16}$$

Finally, given two complex numbers $z_1$ and $z_2$, we define the multiplication as

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) && (17)\\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \in \mathbb{C} && (18) \end{aligned}$$

**Exercise 11.** *Solve the following equations:*

- $z^2 = 1 + i$
- $z^2 + (\sqrt{3} + i)z + 1 = 0$
- *Find the cubic roots of* $1$
- *Find a solution to the equation* $x^2 + 1 = 0$
- *Find a solution to the equation* $x^4 + 1 = 0$
- *Find a solution to the equation* $x^2 + 2x + 2 = 0$
- *Try to expand and simplify* $(\cos(2\pi/3) + i\sin(2\pi/3))^3$

**Exercise 12.** *Describe the region of the complex plane corresponding to each of the following cases*

- $|z - 2| = 3$
- $|z + i| < 1$
- $|z - 2| - |z + 2| < 2$
- $0 < \mathrm{Re}\left\{iz\right\} < 1$
- $|z - 1 + 2i| > 3$
- $\mathrm{Re}\left\{\frac{z-2}{z-1}\right\} = 0$

## 1.5 Polar representation

To determine the argument of $z$ from the real and imaginary parts, note that from the geometric representation, we have

i) $\theta = \arctan y/x + k\pi,\ k = \begin{cases} 0 & x > 0, y \geq 0 \\ 1 & x < 0 \\ 2 & x > 0, y < 0 \end{cases}$ if $x \neq 0$

ii) $\theta = \begin{cases} \pi/2 & y > 0 \\ 3\pi/2 & y < 0 \end{cases}$ if $x = 0$

The representation $z = r\cos\theta\boldsymbol{e}_x + r\sin\theta\boldsymbol{e}_y$ is known as the *polar representation* of the number $z$.

**Exercise 13.** *Find the polar representation of the following numbers:*

- $z = -1 - i$
- $z = 2 + 2i$
- $z = -1 + \sqrt{3}i$
- $z = 1 - i\sqrt{3}$
- $z = -3i$
- $z = 2$
- $z = 1 + \cos a + i\sin a,\ a \in (0, 2\pi)$

For a complex number $z = x + iy$, the number $\overline{z} = x - iy$ is called the complex conjugate of $z$. The definition of the complex conjugate implies the following properties

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(ii) $\overline{z_1 z_2} = \overline{z_1 z_2}$

(iii) For every nonzero $z$, the relation $\overline{z^{-1}} = (\overline{z})^{-1}$ holds. In particular, we have for $z_2 \neq 0$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \tag{19}$$

(iv) Finally for all $z \in \mathbb{Z}$, $\text{Re}\{z\} = \frac{z+\overline{z}}{2}$, $\text{Im}\{z\} = \frac{z-\overline{z}}{2i}$

Note that to obtain the inverse of a complex number $z \in \mathbb{C}$, one can rely on the conjugate as

$$\frac{1}{z} = \frac{\overline{z}}{z - \overline{z}} = \frac{x - iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$$

Similarly, we have

$$\frac{z_1}{z_2} = \frac{z_1\overline{z}_2}{z_2\overline{z}_2} = \frac{(x_1+y_1 i)(x_2 - iy_2)}{x_2^2+y_2^2} = \frac{x_1x_2 + y_1y_2}{x_2^2+y_2^2} + \frac{-x_1y_2 + x_2y_1}{x_2^2+y_2^2}i$$

**Exercise 14.** *Let $z_1, z_2 \in \mathbb{C}$, prove that the number $n = z_1\overline{z_2} + \overline{z_1}z_2$ is a real number.*

**Exercise 15.** *Prove the following properties $|z_1z_2| = |z_1||z_2|$, $|z_1/z_2| = |z_1|/|z_2|$, $|z^{-1}| = |z|^{-1}$, $z \neq 0$.*

**Exercise 16.** *Provide the algebraic form for the following complex number*

$$z = \frac{5+5i}{3-4i} + \frac{20}{4+3i} \tag{20}$$

**Exercise 17.** *Prove the identity $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2 + |z_2|^2)$*

**Exercise 18.** *Prove that if $|z_1| = |z_2| = 1$ and $z_1z_2 \neq -1$ then $\frac{z_1+z_2}{1+z_1z_2}$ is a real number*

**Exercise 19.** *Prove that $\sqrt{3} \leq |1+z| + |1+z+z^2| \leq 13/4$ for all complex numbers such that $|z| = 1$*

**Exercise 20.** *Compute the following numbers*

1. $(2-i)(-3+2i)(5-4i)$
2. $\left(\frac{1+i}{1-i}\right)^{16} + \left(\frac{1-i}{1+i}\right)^{8}$
3. $\left(\frac{-1+i\sqrt{3}}{2}\right) + \left(\frac{1-i\sqrt{7}}{2}\right)^{6}$

**Exercise 21.** *Find the polar representation for the following numbers*

1. $z_1 = -1 - i$
2. $z_2 = 2 + 2i$
3. $z_3 = -1 + i\sqrt{3}$

**Proposition 1.** *Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then*

$$z_1z_2 = r_1r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right) \tag{21}$$

**Exercise 22.** *Provide the algebraic form for the following complex number*

$$z = \frac{5+5i}{3-4i} + \frac{20}{4+3i} \tag{22}$$

**Exercise 23.** *Prove the identity* $|z_1 + x_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

**Exercise 24.** *Prove that if* $|z_1| = |z_2| = 1$ *and* $z_1 z_2 \neq -1$*, then* $\dfrac{z_1 + z_2}{1 + z_1 z_2}$ *is a real number*

**Exercise 25.** *Prove that* $\sqrt{3} \leq |1 + z| + |1 - z + z^2| \leq 13/4$ *for all complex numbers such that* $|z| = 1$

**Exercise 26.** *Compute the following complex number*

- $(2 - i)(-3 + 2i)(5 - 4i)$
- $\left(\frac{1+i}{1-i}\right)^{16} + \left(\frac{1-i}{1+i}\right)^{8}$
- $\left(\frac{-1+i\sqrt{3}}{2}\right) + \left(\frac{1-i\sqrt{7}}{2}\right)^{6}$

**Exercise 27.** *Find the polar representation for the following numbers*

- $z_1 = -1 - i$
- $z_2 = 2 + 2i$
- $z_3 = -1 + i\sqrt{3}$

**Proposition 2.** *Let* $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$*,* $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$*. Then*

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right) \tag{23}$$

Using induction, extend the formula (21) to the relation below

$$\prod_{k=1}^{m} z_k = \prod_{k=1}^{m} r_k \left(\cos\sum_{k=1}^{m}\theta_k + i\sin\sum_{k=1}^{m}\theta_k\right)$$

**Proposition 3** (De Moivre)**.** *For* $z = r(\cos\theta + i\sin\theta)$ *and* $n \in \mathbb{N}$ *we have*

$$z^n = r^n(\cos n\theta + i\sin n\theta) \tag{24}$$

**Exercise 28.** *Simplify the following expression*

$$z = \frac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}} \tag{25}$$

**Exercise 29.** *Find* $|z|$ *and* $\arg\{z\}$ *for the following complex numbers*

- $z = \dfrac{(2\sqrt{3}+2i)^8}{(1-i)^6} + \dfrac{(1+i)^6}{(2\sqrt{3}-2i)^8}$
- $z = (1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n$

Let $\theta$ be a real number, we define the following complex extension for the exponential

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{26}$$

More generally if $z = x + iy$, we define $e^z$ as $e^z = e^x e^{iy}$. It is a natural extension to use as it agrees with Taylor's formula (i.e. since we defined the power of imaginary numbers, any polynomial can be extended to the complex numbers and so can $e^x, \cos x, \sin x$ since those functions are real analytic with an infinite radius of convergence). Recall that we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{27}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \tag{28}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \tag{29}$$

Note that

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} + i\frac{\theta^3}{3!} - \frac{\theta^4}{4!} + \dots \tag{30}$$

$$= \cos\theta + i\sin\theta \tag{31}$$

Note that from this notation, it becomes easy to prove de Moivre's formula.

**Exercise 30.** *Express the following complex numbers in modulus-argument form*

- $z = (1+i)(1+i\sqrt{3})(\sqrt{3}-i)$
- $z = \dfrac{(1+i)^5(1-i\sqrt{3})^5}{(\sqrt{3}+i)^4}$

## 1.6 Convergence of sequences

**Definition 1.** *A function $f : \mathbb{N} \mapsto X$ whose domain of definition is the set of natural numbers is called a* sequence. *The values $x_n = f(n)$ of the function are called the* terms *in the sequence.*

**Definition 2.** *A number $\alpha \in \mathbb{R}$ is called the* limit *of a numerical sequence $\{x_n\}$ if for every neighborhood $V(\alpha)$ there exists an index $N$ depending on $V(\alpha)$ such that all the terms in the sequence having index larger than $N$ belong to the neighborhood $V(\alpha)$. Mathematically, we can write this as*

$$\forall \varepsilon > 0, \quad \text{there exists an index } N \text{ s.t. } |x_n - A| < \varepsilon \quad \text{for all } n > N$$

**Definition 3.** *A sequence $\{x_n\}$ is called a fundamental or Cauchy sequence if for any $\varepsilon > 0$, there exists an index $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ whenever $m, n > N$*

**Theorem 1.** *A numerical sequence converges if an only if it is a Cauchy sequence*

## 1.7 Analyticity, continuity, limits, etc..

A function $f$ which can be locally approximated by a polynomial is called *real analytic.*

**Definition 4.** *Let $E$ be a subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$ be a function. If $a$ is an interior point of $E$, we say that $f$ is real analytic at $a$ if there is an open interval $(a-r, a+r)$ in $E$ for some $r > 0$ such that there exists a power series*

$$\sum_{n=0}^{\infty} c_n (x-a)^n \tag{32}$$

*centered at $a$ which has radius of convergence greater than or equal to $r$ and which converges to $f$ on $(a-r, a+r)$. If $E$ is an open set, and $f$ is real analytic at every point $a$ of $E$, we say that $f$ is real analytic on $E$.*

**Proposition 4.** *Real analytic functions are $k$-times differentiable for every $k \geq 0$*

**Proposition 5** (Taylor's formula, such as appearing in [1])**.** *Let $E$ be a subset of $\mathbb{R}$. Let $a$ be an interior point of $E$, and let $f : E \to \mathbb{R}$ be a function which is real analytic at $a$ and has the power series expansion*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \tag{33}$$

*for all $x \in (a-r, a+r)$ and $r > 0$. Then for any integer $k \geq 0$, we have*

$$f^{(k)}(a) = k!\, c_k \tag{34}$$

*where $k! = 1.2.3.4 \ldots k$ (and we adopt the convention $0! = 1$).*

**Definition 5** (Limit of a function)**.** *Let $f : E \to \mathbb{R}$ be a real valued function defined on $E$. We will say that a function $f : E \to \mathbb{R}$ tends to $L$ as $x$ tends to $x_0$ or that $L$ is the limit of $f$ as $x$ tends to $x_0$, if for every $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ for every $x$ such that $0 < |x - x_0| < \delta$. Mathematically we have*

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall\, x \in E \; (0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon) \tag{35}$$

**Exercise 31.** *Using the definition of the limit, show that on $E = \mathbb{R} \setminus \{0\}$, the limit*

$$\lim_{\substack{x \to 0 \\ x \in E}} x \sin\left(\frac{1}{x}\right) = 0 \tag{36}$$

**Exercise 32.** *Show that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ (hint: use the trigonometric circle and show that $\cos^2 x < \dfrac{\sin x}{x} < 1$ for $0 < |x| < \pi/2$)*

**Exercise 33.** *Consider the function*

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \tag{37}$$

*what is the limit*

$$\lim_{\substack{x \to 0 \\ x \in \mathbb{R} \setminus \{0\}}} f(x) \tag{38}$$

*how about $\lim_{\substack{x \to 0 \\ x \in \mathbb{R}}} f(x)$?*

For a limit to exist and equal $L$, the function $f(x)$ has to be approximately equal to $L$ on both sides of $x = a$. I.e.

$$\lim_{x \to a} f(x) = L \quad \text{iff} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \tag{39}$$

Here is what the limit can be

- $\lim_{x \to a} f(x) = L = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ (the only case in which the limit exists)
- $\lim_{x \to a} f(x) = +\infty$ (the notation $\infty$ here means that $f(x)$ eventually becomes larger than any real number as $x$ gets closer and closer to $a$)
- $\lim_{x \to a} f(x) = -\infty$ (In this case, the notation $-\infty$ means that as $x$ gets closer and closer to $a$, the value of $f$ becomes more negative than any *real* number. As for the $+\infty$ case, the limit therefore does not exist either)

Generally speaking, keep in mind that $\infty$ is not a *real* quantity (i.e. one that can be observed in nature). In the last two cases above, it is therefore also appropriate to conclude that the limit does not exist.

**Exercise 34.** *Find the following limits*

- $\lim\limits_{x \to 1} \dfrac{x-1}{x^2-1}$
- $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2-9}-3}{t^2}$

**Exercise 35.** *Find the following limits*

- $\lim\limits_{x \to 3^+} \dfrac{2x}{x-3}$
- $\lim\limits_{x \to 3^-} \dfrac{2x}{x-3}$

**Exercise 36.** *Estimate the value of the following limit*

$$\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}$$

**Proposition 6** (Properties of the limit)**.** *Let $c$ be a constant and let us assume that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then*

- $\lim\limits_{x \to a} [f(x)+g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$
- $\lim_{x\to a} [f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim\limits_{x \to a} [cf(x)] = c \lim\limits_{x \to a} f(x)$
- $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
- $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim\limits_{x \to a} g(x) \neq 0$
- $\lim\limits_{x \to c} f(g(x)) = f(\lim\limits_{x \to c} g(x))$ *if $f$ is continuous at* $\lim_{x\to c} g(x)$

We now state a few additional properties which can reveal helpful when computing practical limits.

**Theorem 2.** *If $f(x) \leq g(x)$ when $x$ is near $a$ and the limits of $f$ and $g$ both exist as $x$ approach $a$, then*

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \tag{40}$$

**Theorem 3** (Squeeze theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ when $x$ is near $a$ and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

*then*

$$\lim_{x \to a} g(x) = L$$

**Exercise 37.** *Using Theorem 3, find the limit of $x^2 \sin 1/x$ at 0.*

Recall that the derivative can also be defined through the notion of limit. I.e.

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

This definition can sometimes be used to calculate the a limit as shown by the following examples

$$\lim_{x\to\pi}\frac{\cos(x)+1}{x-\pi}=-\sin(\pi)=0,\quad \lim_{x\to 0}\frac{e^{3x}-1}{x}=3e^{3\times 0}=3$$

**Exercise 38.** *Using the definition of the derivative (or any other approach), find the following limits*

- $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x}$
- $\lim\limits_{x\to 0}\dfrac{\log(1+2x)}{\sin(4x)}$
- $\lim\limits_{x\to 1}\dfrac{\ln x}{x-1}$
- $\lim\limits_{x\to+\infty}x\ln\left(1+\dfrac{2}{x}\right)$

## 1.8 Limits at infinity

**Exercise 39.** *Draw the function* $f(x)=\dfrac{x^2-1}{x^2+1}$ *and find the limit* $\lim_{x\to\infty}f(x)$*.*

**Definition 6.** *Let $f$ be a function defined on the interval $(a,+\infty)$ the notation $\lim_{x\to\infty}f(x)=L$ means that the value of $f$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large.*

**Definition 7.** *As for $+\infty$, for a function $f$ defined on some interval $(-\infty,a)$, $\lim\limits_{x\to-\infty}f(x)=L$ means that the values of $f$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large and negative*

**Theorem 4.** *For any rational number $r>0$ such that $x$ is well defined for all $x$, we have*

$$\lim_{x\to+\infty}\frac{1}{x^r}=\lim_{x\to-\infty}\frac{1}{x^r}=0$$

**Exercise 40.** *Use the previous theorem to deduce that for any two polynomials $p$ and $q$ such that deg($p$) = deg($q$) with $p(x)=p_nx^n+r(x)$, $q(x)=q_nx^n+s(x)$, we have*

$$\lim_{x\to\infty}\frac{q(x)}{q(x)}=\frac{p_n}{q_n}$$

*Use the previous result to derive the following limits*

- $\lim\limits_{x\to\infty}\dfrac{3x^2-x-2}{5x^2+4x+1}$
- $\lim\limits_{x\to\infty}\dfrac{3x^2-x-2}{5x^2+4x+1}$
- $\lim\limits_{x\to+\infty}\dfrac{\sqrt{2x^2+1}}{3x-5}$
- $\lim\limits_{x\to-\infty}\dfrac{\sqrt{2x^2+1}}{3x-5}$

- $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

**Exercise 41.** *Find the limit* $\lim_{x\to\infty}\left(x^2-x\right)$. *(note that the limit of a difference is the difference of the limits only if both limits exist)*

**Exercise 42.** *Find the limit*

$$\lim_{x\to\infty}\frac{x^2+x}{3-x}$$

**Exercise 43.** *Find the following limits*

- $\lim_{x\to\infty}\frac{2e^x}{e^x-5}$
- $\lim_{x\to\pm\infty}\frac{1+x^4}{x^2-x^4}$
- $\lim_{h\to 0}\frac{\sqrt[4]{16+h}-2}{h}$
- $\lim_{x\to\infty}\frac{x^3+5x}{2x^3-x^2+4}$
- $\lim_{x\to 1}e^{x^3-x}$
- $\lim_{x\to 5}\frac{2^x-32}{x-5}$
- $\lim_{v\to 4^+}\frac{4-v}{|4-v|}$
- $\lim_{t\to 1}\frac{t^4+t-2}{t-1}$

We have already seen that when we need to compute the limit of a ratio, it might happen (for example in the case of a ratio of polynomials) that despite the fact that the limit of the numerator and denominator do not exist, the limit of the quotient could itself be determined (for example by defining it to the the ratio of the leading coefficients). In general, if we have a limit of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

where both $\lim_{x\to a}f(x)=\lim_{xa}g(x)=0$ or $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=\pm\infty$, the limit of the ratio may or may not exist and is called an *indeterminate form.* Sometimes as we saw the limit can be derived through the use of appropriate arguments (e.g. ratio of leading coefficients as said above, geometric arguments or the squeeze theorem). Sometimes however, it is not obvious how to evaluate the limit (this happens in particular when there is a struggle between the numerator and the denominator). A general approach when the previous arguments failis to turn to the mean value theorem. If $f$ is a continuous function on the closed interval $[a,b]$ and differentiable on the open interval $(a,b)$ then there exists a $c\in(a,b)$ s.t.

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

From this, provided that both $f$ and $g$ are differentiable (hence continuous) and $g'(x)\neq 0$ on an open interval $I$ that surrounds $a$ (note that $a$ does not have to be included), we have for $c,d\in[x,a)$

$$\lim_{x\to a}\frac{f'(c)}{g'(d)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

Now assuming $\lim_{x\to a}f(x)=f(a)=0=\lim_{x\to a}g(x)=g(a)=0$

$$\lim_{x\to a}\frac{f'(c)}{g'(d)}=\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{\substack{c\to a\\ d\to a}}\frac{f'(c)}{g'(d)}=\frac{f'(a)}{g'(a)}$$

One can derive a similar proof for the case $+\infty/+\infty$ and $-\infty/-\infty$. If we let $\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$ then for all $\delta>0$ there exists $\beta\in(a,b)$ s.t. $\forall x\in(a,\beta)$

$$\left|\frac{f'(x)}{g'(x)}-L\right|<\delta \tag{41}$$

let $x, y \in (a, \beta)$, $x < y$ since we assumed that the functions $f$ and $g$ are continuous and differentiable on $[x, y]$, we can again use the mean value theorem. There exist $c, d \in [x, y]$ s.t.

$$\frac{f'(c)}{g'(d)} = \frac{f(x) - f(y)}{g(x) - g(y)}$$

From (41) we clearly have

$$\left| \frac{f(x) - f(y)}{g(x) - g(y)} - L \right| < \delta$$

Rearranging, we get

$$\left| \frac{f(x)/g(x) - f(y)/g(x)}{1 - g(y)/g(x)} - L \right| < \delta \tag{42}$$

Then, assuming without loss of generality that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = +\infty$ and keeping $y$ fixed, we get $f(y)/g(x) \to 0$ as well as $g(y)/g(x) \to 0$. substituting this into (42), for $x$ sufficiently close to $a$, we have

$$|f(x)/g(x) - L| < \delta$$

We can now summarize our findinds through the following proposition known as *l'Hospital's rule*

**Proposition 7.** *Suppose that $f$ and $g$ are differentiable and $g'(x) \neq 0$ on an open interval $I$ that contains $a$ (except possibly at $a$). Suppose that*

$$\begin{aligned} & \lim_{x \to a} f(x) = 0 \quad \textit{and} \quad \lim_{x \to a} g(x) = 0 \\ \textit{or} \quad & \lim_{x \to a} f(x) = \pm\infty \quad \textit{and} \quad \lim_{x \to a} g(x) = \pm\infty \end{aligned}$$

*then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \textit{provided that this limit exists}$$

**Exercise 44.** *Find the following limits:*

- $\displaystyle\lim_{x \to 1} \frac{\ln x}{x - 1}$
- $\displaystyle\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$
- $\displaystyle\lim_{x \to \infty} \frac{e^x}{x^2}$
- $\displaystyle\lim_{x \to 0} \frac{\tan x - x}{x^3}$

Just as for $\infty/\infty$ and $0/0$, when $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \pm\infty$, then again it is not clear what the value of $\lim_{x \to a} f(x)g(x)$ will be. One way to handle such an indeterminate product is to rewrite it as

$$f(x)g(x) = \frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x)g(x) = \frac{g(x)}{1/f(x)}$$

this turns the limit into an undeterminate form of the type $0/0$ or $\pm\infty/\pm\infty$ which can be simplified using l'Hospital's rule.

Similarly when we have the limit of a difference

$$\lim_{x \to a} [f(x) - g(x)]$$

and $\lim_{x \to a} f(x) = +\infty$ as well as $\lim_{x \to a} g(x) = +\infty$, there is again a competition between the two limits and we end up with an indeterminate form. In this case, one can again make use of l'Hospital's rule by first transforming the difference into a quotient and then apply the rule to the resulting $0/0$ or $\infty/\infty$ form.

### 1.8.1 Indeterminate powers

Another setting that can give rise to an indeterminate form is the limit of a power. In this case we have

$$\lim_{x\to a} [f(x)]^{g(x)}$$

where the limits of $f$ and $g$ take any of the following values:

- $\lim_{x\to a} g(x) = 0,\ \lim_{x\to a}$. In this case, the indeterminate is of the form $0^0$
- $\lim_{x\to a} f(x) = \infty,\ \lim_{x\to a} g(x) = 0$. In this case, the indeterminate is of the form $\infty^0$
- $\lim_{x\to a} f(x) = 1,\ \lim_{x\to a} g(x) = \pm\infty$. In this case the indeterminate is of the type $\alpha^\infty$ with $\alpha \to 1$.

The three cases above can be treated by taking the logarithm. Note that if we let $y = \lceil f(x)\rceil^{g(x)}$, then we have $\ln y = g(x)\ln f(x)$ which is now of the form $0 \times \pm\infty$ and can be treated, for example, by l'Hospital.

**Exercise 45.** *Calculate the following limits*

- $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$
- $\lim_{x\to 0} x^x$
- $\lim_{x\to\infty} x^3 e^{-x^2}$
- $\lim_{x\to 0} (1-2x)^{1/x}$
- $\lim_{x\to 2} \dfrac{x^2+x-6}{x-2}$
- $\lim_{x\to\infty} \left(1+\dfrac{a}{x}\right)^{bx}$
- $\lim_{x\to\infty} \dfrac{\ln\ln x}{x}$
- $\lim_{x\to\infty} (e^x+x)^{1/x}$
- $\lim_{x\to 0} \dfrac{e^x-1-x}{x^2}$
- $\lim_{x\to 0} \dfrac{\cos x - 1 + x^2/2}{x^4}$
- $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

**Exercise 46.** *Prove that* $\lim_{x\to\infty} \dfrac{e^x}{x^n}$ *for any positive integer* $n$*. This shows that the exponential function grows faster than any power of* $x$

**Exercise 47.** *What happens if you try l'Hospital's rule to evaluate*

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \tag{43}$$

**Exercise 48.** *Determine the following limits*

- $\lim_{x\to -3^+} \dfrac{1}{-2x-6}$
- $\lim_{x\to 3^+} \dfrac{1-4x}{x-3}$
- $\lim_{x\to +\infty} \dfrac{\sqrt{x}+2-3x}{x}$
- $\lim_{x\to 0^+} \left(\left(1+\dfrac{1}{\sqrt{x}}\right)(x-3)\right)$
- $\lim_{x\to -\infty} \dfrac{2x+5}{\sqrt{-x}}$
- $\lim_{x\to -2^-} \dfrac{-2x}{3x+6}$

### 1.9 Exponential and Compounding

A *Compound interest* is an interest that is earned on previous interests. As an example, if you invest 300€ in an interest fund for two years at 10% interest annually, you will earn 30 € the first year but then you will earn 10% of 330 € for the second year, that is to say a total of 63€ in interest. Mathematically the compound interest formula reads as

$$A = A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

where $A$ is the final amount of money you get from your investment, $A_0$ is the original investment (a.k.a principal amount), $r$ is the interest rate, $n$ is the number of compounding per year (i.e. how many times a year do you reinvest your gains) and $t$ is the total duration of the investment.

**Exercise 49.** *Assuming a* 0.08 *(i.e.* 8%*) interest rate and an initial investment of* 2500 € *, how much will you gain after* 40 *years if you compound once a year? twice a year? every month? every day?*

Stocks do not generate compound interests but they have compounding values. Let's say you invest 1000 € in a $S\&P$ index fund, with an average annual return of 9% and little variance. Next year, your shares will be worth 1090 € and so on. Alternatively, if you can put aside 5€ a day into an account with 10% annual return, it could get you around $2.3M$ € in 50 years!

Finally, a last way to implement compounding is through *dividend reinvestment*. Several companies offer dividends on their stocks which are distributed as cash to the shareholders. Such dividends can be reinvested thus implementing compounding. Moreover, some companies even offer dividend reinvestment plans which allow investors to automatically reinvest the cash they receive as dividends to purchase additional shares.

The compounding equation is in fact connected to the definition of the exponential function. To connect the notion of compounding to the exponential function, we assume that reinvestment is done $n$ times a year at a $x$% rate and we consider the result of compouning for increasinly large values of $n$, i.e.

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{n} \tag{44}$$

The limit (44) defines the exponential function $e^x$.

## 2 Integration

Recall that given a continuous function $f$, the area of the region $S$ that lies under the graph of $f$ is the limit of the areas of the approximating rectangles.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + \dots f(x_n)\Delta x\right]$$

where the $x_1, x_2, \dots$ are defined as a partition of $[a, b]$ (see Fig 4).

Note that as we take the limit of an increasingly large partition, the area under the graph of $f(x)$ can

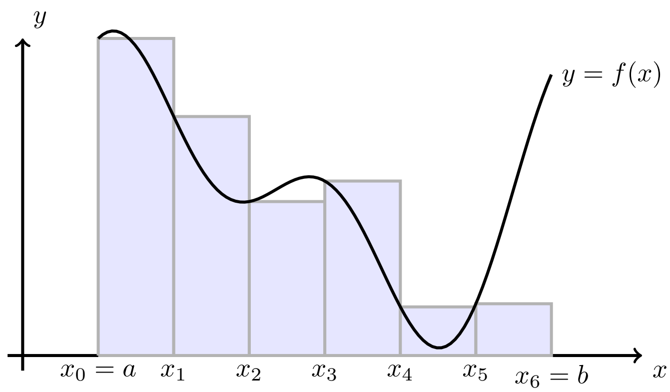



Figure 4: Riemann Sum

be defined through any of the following partitions

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \end{aligned}$$

where $x_i^*$ is any value between $x_{i-1}$ and $x_i$. Formally we define the *Riemann integral* as follows. For a given interval $[a,b] \subseteq \mathbb{R}$, and a given partition $a = x_0 < x_1 < \ldots < x_n = b$, together with a collection $\{t_i\}_{i=1}^{n}$ of values satisfying $x_{i-1} \leq t_i \leq x_i$, $1 \leq i \leq n$, we further define the stepsize of the partition as the largest interval from the partition. I.e.

$$\max\{x_i - x_{i-1} \mid 1 \leq i \leq n\}$$

We let $\Sigma^*(p)$ to denote the set of all partitions whose stepsize is smaller than or equal to $p$. For a continuous function $f$, if the limit

$$\lim_{\substack{\Sigma^*(p) \\ p\to 0}} \sum_{i=1}^{n} (x_i - x_{i-1}) f(t_i) \tag{45}$$

converges, the limit is called the *Riemann integral* of the function $f$ and written

$$\int_a^b f(t)\, dt$$

## 2.1 Properties of the integral

- $\displaystyle\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$
- $\displaystyle\int_a^a f(x)\, dx = 0$
- $\displaystyle\int_a^b c\, dx = c(b-a)$ for any constant $c \in \mathbb{R}$

- $\displaystyle\int_a^b [f(x)+g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
- $\displaystyle\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$ for any constant $c \in \mathbb{R}$
- $\displaystyle\int_a^b [f(x)-g(x)]\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$
- $\displaystyle\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$

**Theorem 5** (Fundamental Theorem of Calculus (Part I)). *If $f$ is continuous on $[a,b]$ then the function $g$ defined by*

$$g(x) = \int_a^x f(t)\ dt, \quad a \leq x \leq b$$

*is continuous on $[a,b]$ and differentiable on $(a,b)$. Moreover, $g'(x) = f(x)$.*

**Exercise 50.** *Find the derivative of the function*

$$g(x) = \int_0^x \sqrt{1+t^2}\ dt$$

**Exercise 51.** *Using the chain rule together with Theorem 5, find the following derivative*

$$\frac{d}{dx}\int_1^{x^4} \sec t\ dt$$

**Theorem 6** (Fundamental Theorem (Part II)). *if $f$ is continuous on $[a,b]$ then $\int_a^b f(x)\ dx = F(b) - F(a)$ where $F$ is any primitive of $f$ (that is any function such that $F'(x) = f(x)$).*

What the fundamental theorem tells us is that to evaluate the itegral of a function, it suffices to ask oneself the question: *What is the function $F$ whose derivative is equal to the integrand $f(x)$?*

**Exercise 52.** *Evaluate the following integrals:*

- $\displaystyle\int_3^6 \frac{dx}{x}$
- $\displaystyle\int_{-1}^3 \frac{1}{x^2}\ dx$
- $\displaystyle\int_1^3 e^x\ dx$

**Exercise 53.** *Using the fundamental theorem of calculus, evaluate the derivatives of the following functions*

- $\displaystyle g(y) = \int_2^y t^2 \sin t\ dt$
- $\displaystyle g(x) = \int_1^{\cos x} (1+v^2)^{10}\ dv$
- $\displaystyle g(x) = \int_{1-3x}^1 \frac{u^3}{1+u^2}\ du$

**Exercise 54.** *Evaluate the following Riemann integrals:*

- $\displaystyle\int_{-1}^2 (x^3-2x)\ dx$
- $\displaystyle\int_\pi^{2\pi} \cos\theta d\theta$

- $\displaystyle\int_1^9 \frac{1}{2x}\,dx$
- $\displaystyle\int_1^2 \frac{4+u^2}{u^3}\,du$
- $\displaystyle\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta$
- $\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}}\,dt$
- $\displaystyle\int_1^2 (1+2y)^2\,dy$
- $\displaystyle\int_{-2}^2 f(x)\,dx$
  where $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4-x^2 & \text{if } 0 < x \le 2 \end{cases}$

## 2.2 Common integrals

- $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$
- $\displaystyle\int [f(x)+g(x)]\,dx = \int f(x)dx + \int g(x)dx$
- $\displaystyle\int k\,dx = kx + C$
- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq 1)$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int a^x dx = \frac{a^x}{\ln a} + C$
- $\displaystyle\int \sin x dx = -\cos x + C$
- $\displaystyle\int \cos x dx = \sin x + C$
- $\sec^2 x dx = \tan x + C$
- $\displaystyle\int \csc^2 x dx = -\cot x + C$
- $\displaystyle\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
- $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
- $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq 1)$
- $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$
- $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$
- $\displaystyle\int \sec x\tan x dx = \sec x + C$
- $\int cscx \cot x dx = -\csc x + C$

Finally, if we define the hyperbolic sine (sinh) and cosine (cosh) as

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

we also have

- $\displaystyle\int \sinh x dx = \cosh x + C$
- $\displaystyle\int \cosh x dx = \sinh x + C$

**Exercise 55.** *Prove the two formulas above (*sinh *and* cosh*).*

**Exercise 56.** *Evaluate the following integrals or find the primitive*

- $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}d\theta$
- $\displaystyle\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx$
- $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}dt$
- $\displaystyle\int \frac{\sin 2x}{\sin x}dx$

- $\int (1 + \tan^2 x) dx$
- $(\csc^2 t - 2e^t) dt$
- $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta \tan^2\theta}{\sec^2\theta} d\theta$
- $\int_1^4 \sqrt{\frac{5}{x}} dx$
- $\int_1^{64} \frac{1 + \sqrt[3]{x}}{\sqrt{x}} dx$
- $\int_0^{1/\sqrt{3}} \frac{t^2 - 1}{t^4 - 1} dt$
- $\int_{-1}^{2} (x - 2|x|) dx$

## 2.3 Substitution, integration by parts, etc...

### 2.3.1 The substitution rule

So far we have focused on a limited set of simple primitives. When facing more complicated expressions, such (46) below,

$$\int 2x\sqrt{1 + x^2} dx \tag{46}$$

we need more advanced techniques. We will cover two such techniques which are particularly useful when facing complicated primitives: *the substitution rule* and *integration by parts*. We will then discuss a couple of other helpful tricks.

Let us first go back to (46). Note that if we let $u(x)$ to denote the function $u(x) = 1 + x^2$, using the chain rule , we have $u'(x) = 2x$ and the integral (46) can thus read as

$$\int 2x\sqrt{1 + x^2} dx = \int u'(x) u^{1/2}(x) dx \tag{47}$$

On the other hand, using the chain rule, if we take the derivative of the function $u(x)^{3/2}$, we get

$$\frac{d}{dx} u^{3/2}(x) = \frac{3}{2} u'(x) u^{1/2}(x) = \frac{3}{2} 2x(1 + x^2)^{1/2}$$

which is (up to a constant factor) precisely the integrand from (46). From this, using the fundamental theorem, we can write

$$\int 2x\sqrt{1 + x^2} = \int u'(x) u^{1/2}(x) dx = \frac{2}{3} u^{3/2}(x)$$

This idea reads generally as

$$\int u'(x) f'(u(x)) dx = f(u(x)) \tag{48}$$

and is sometimes known as the *substitution rule*.

The substitution rule can be applied to definite integrals as well, provided that the integration bounds are modified appropriately. In this case, we have

$$\int_a^b f(u(x)) u'(x) dx = \int_{u(a)}^{u(b)} f(u) du \tag{49}$$

**Exercise 57.** *Compute the following integral through substitution*

- $\displaystyle\int \frac{(\ln x)^2}{x} dx$
- $\displaystyle\int \frac{\cos\sqrt{t}}{\sqrt{t}}$
- $\displaystyle\int \frac{e^x}{e^x+1} dx$
- $\displaystyle\int x\sin(x^2)dx$
- $\displaystyle\int (3x-2)^{20}dx$
- $\displaystyle\int \frac{dx}{5-3x}$
- $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}\sin^{-1} x dx$
- $\displaystyle\int e^x\sin(e^x)dx$
- $\displaystyle\int \frac{dx}{ax+b} dx \;\; a\neq 0$
- $\displaystyle\int \frac{\cos x}{\sin^2 x} dx$
- $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$
- $\displaystyle\int_0^a x\sqrt{a^2-x^2}dx$

**Exercise 58.** *If $a$ and $b$ are positive numbers, show that*

$$\int_0^1 x^a(1-x)^b dx = \int_0^1 x^b(1-x)^a dx$$

Another important rule is *integration by parts*. When studying derivatives, you probably learned that, for any product of two functions $f(x)g(x)$, you could write

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

Integrating both sides of this equation, we get

$$\int \left[f(x)g'(x) + g(x)f'(x)\right] dx = f(x)g(x) \tag{50}$$

Equation (50) can be rearranged as

$$\int f(x)g'(x)dx = f(x)g(x) - \int_g (x)f'(x)dx \tag{51}$$

which is known as the *integration by parts* formula. A similar formula can be derived for definite integrals. I.e.

$$\int_a^b f(x)g'(x)dx = \left. f(x)g(x)\right|_a^b - \int_a^b g(x)f'(x)dx \tag{52}$$

**Exercise 59.** *Evaluate the following integrals:*

- $\displaystyle\int (\ln x)^2 dx$
- $\displaystyle\int_0^1 (x^2+1)e^{-x}dx$
- $\displaystyle\int e^{2\theta}\sin 3\theta d\theta$
- $\displaystyle\int x^2\cos mx dx$
- $\displaystyle\int t\sin 2t dt$
- $\displaystyle\int x^3\cos x dx$
- $\displaystyle\int_0^{1/2} \cos^{-1}(x)dx$
- $\displaystyle\int \sin(\ln x)dx$
- $\displaystyle\int \cos\sqrt{x}dx$

## 2.4 Integration of rational functions

Another important setting that frequently arises in the applied sciences is the integration of *rational functions*, that is functions that are given by a ratio of polynomials. When facing such functions, the first step is to write them as a sum of simpler expressions known as *partial fractions*.

As an example, consider the following integral

$$\int \frac{x+5}{x^2+x-2}dx$$

First note that by factoring the denominator, we can write

$$\frac{x+5}{x^2+x-2} = \frac{2}{x-1} - \frac{1}{x+2}$$

From this we get

$$\int \frac{x+5}{x^2+x-2}dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$

On this second integral, we can then apply classical integration formulas (see section 2.2) to get

$$\int \frac{x+5}{x^2+x-2}dx = 2\log(|x-1|) - \log(|x+2|) + C$$

As we saw above, the first step in integrating rational functions consists in facroring the denominator. Depending on the form of this denominator, we distinguish 4 situations below:

- The first situation that can arise is when the denominator is a product of distinct linear factors. That is to say, for a rational function of the form $P(x)/Q(x)$, we have

  $$Q(x) = (a_1x+b_1)(a_2x+b_2)\dots(a_nx+b_n)$$

  The first step in this case consists in finding a decomposition of the form

  $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_n}{a_nx+b_n}$$

  To determine the coefficients of the decomposition, one can either multiply both sides of the equation by the product of the denominators and identify the coefficients or one can note that, after multiplying by the product of the denominators and getting the right-hand side

  $$\begin{aligned} &A_1\frac{(a_2x+b_2)\dots(a_nx+b_n)}{\prod_{i=1}^n(a_ix+b_i)} + \dots + A_n\frac{(a_1x+b_1)\dots(a_{n-1}x+b_{n-1})}{\prod_{i=1}^n(a_ix+b_i)} \\ &= \prod_{i=1}^n \frac{1}{(a_ix+b_i)}\left[A_1(a_2x+b_2)\dots(a_nx+b_n) + A_2(a_1x+b_1)(a_3x+b_3)\dots(a_nx+b_n)\right. \\ &\qquad \left. + \dots + A_n(a_1x+b_1)\dots(a_{n-1}x+b_{n-1})\right] \end{aligned} \tag{53}$$

  From this, mutiplying both sides of the equality (53) (which is authorized even if some of the factors vanish as it is a multiplication) by $\prod_{i=1}^n(a_ix+b_i)$, we get the equation

  $$\begin{aligned} P(x) = &\left[A_1(a_2x+b_2)\dots(a_nx+b_n) + A_2(a_1x+b_1)(a_3x+b_3)\dots(a_nx+b_n)\right. \\ &\left. + \dots + A_n(a_1x+b_1)\dots(a_{n-1}x+b_{n-1})\right] \end{aligned} \tag{54}$$

  We can then cancel most of the terms on the right-hand side of (54) by substituting the root of any particular factor for $x$. I.e. if we for example take $x = -b_2/a_2$, the right-hand side in (54) reduces to

  $$A_2\left(a_1\left(-\frac{b_2}{a_2}\right)+b_1\right)\left(a_3\left(-\frac{b_2}{a_2}\right)+b_3\right)\dots\left(a_n\left(-\frac{b_2}{a_2}\right)+b_n\right)$$

and we can then determine $A_2$ by solving

$$P(-\frac{b_2}{a_2}) = A_2\left(a_1\left(-\frac{b_2}{a_2}\right)+b_1\right)\left(a_3\left(-\frac{b_2}{a_2}\right)+b_3\right)\dots\left(a_n\left(-\frac{b_2}{a_2}\right)+b_n\right) \tag{55}$$

- A second situation that might occuris when the polynomial $Q(x)$ at the denominator is a product of factors, some of which are repeated. In this case, assuming for example that the factor $(a_1x+b_1)$ is repeated $r$ times, we have

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \dots + \frac{A_r}{(a_1x+b_1)^r} \tag{56}$$

In this case, one can apply the exact same reasoning as before and substitute $-b_2/a_2$ for $x$. The only difference is that we will now determine $A_1$ by taking the derivative of the numerator multiplied by the product of the factors. I.e., multiplying both sides of a sum of an equality of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_1'}{(a_1x+b_1)^2} + \dots + \frac{A_1^{(r-1)}}{(a_1x+b_1)^r} + \frac{A_2}{(a_2x+b_2)} + \dots$$

by the product of the denominators, and canceling this product, we get

$$\begin{aligned} P(x) = &\left[A_1(a_1x+b_1)^{r-1} + A_1'(a_1x+b_1)^{r-2} + \dots + A_1^{r-1}\right](a_2x+b_2)\dots(a_nx+b_n) && (57)\\ &+ A_2(a_1x+b_1)^r(a_3x+b_3)\dots(a_nx+b_n) + \dots && (58)\\ &+ A_n(a_1x+b_1)^r(a_2x+b_2)\dots(a_{n-1}x+b_{n-1}) && (59) \end{aligned}$$

To determine the value of the coefficients $A_1$ to $A_1^{(r-1)}$, it then suffices to take the derivatives of order 1 to $r-1$ and substitute $-b_1/a_1$ for $x$. All the terms multiplied by $(a_1x+b_1)^r$ will then disappear and we will be left with a set of equations in each of the coefficients $A_1$ to $A_1^{(r-1)}$. As an example, for $A_1^{r-2}$, taking the derivative of (60) with respect to $x$, we get

$$\begin{aligned} P'(x) = &\left[A_1a_1(r-1)(a_1x+b_1)^{r-2} + A_1'a_1(r-1)(a_1x+b_1)^{r-2} + \dots + A_1^{r-2}\right](a_2x+b_2)\dots(a_nx+b_n)\\ &+ \left[A_1(a_1x+b_1)^{r-1} + A_1'(a_1x+b_1)^{r-2} + \dots + A_1^{r-1}\right]\frac{d}{dx}\left[(a_2x+b_2)\dots(a_nx+b_n)\right]\\ &+ A_2ra_1(a_1x+b_1)^{r-1}(a_3x+b_3)\dots(a_nx+b_n) + A_2(a_1x+b_1)^r\frac{d}{dx}\left[(a_3x+b_3)\dots(a_nx+b_n)\right]\\ &+ \dots + A_na_1r(a_1x+b_1)^{r-1}(a_2x+b_2)\dots(a_{n-1}x+b_{n-1})\\ &+ A_n(a_1x+b_1)^r\frac{d}{dx}\left[(a_2x+b_2)\dots(a_{n-1}x+b_{n-1})\right] \end{aligned}$$

Substituting $-b_1/a_1$ for $x$ then gives

$$P'(-b_1/a_1) = A_1^{r-2}\left(a_2\left(-\frac{b_1}{a_1}+b_2\right)\right)\dots\left(a_n\left(-\frac{b_1}{a_1}+b_n\right)r\right) \tag{60}$$

- A third situation that could occur is a denominator that cannot be factored. I.e. assume that the quotient is of the form

$$\frac{Ax+B}{ax^2+bx+c} \tag{61}$$

where $b^2-4ac<0$. In this case, one can either find the complex roots, or we can try to find a factorisation of the form

$$\frac{A_1}{x^2+a_1} + \frac{A_2}{x-a_2} + \dots$$

As an example, consider the following rational function

$$\frac{P(x)}{Q(x)} = \frac{x}{(x-2)(x^2+1)(x^2+4)}$$

such a quotient can be expressed as

$$\frac{x}{(x-1)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

From which the coefficients $A, B, C, D$ and $E$ can be determined.However, note that although we can apply the previous ideas for $A$ for which if we multiply both sides by the product of the denominators and then substitute 2 for $x$ we get

$$A(x^2+1)(x^2+4) + (Bx+C)(x-2)(x^2+4) + (Dx+E)(x^2+1)(x-2) = P(x) = x$$

then

$$A(2^2+1)(2^2+4) = 2,$$

For $B, C, D$ and $E$, we have no choice but to identify the coefficients. Once the coefficients have been determined, we can again rely on classical integration formulas such as

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

- Finally, a last situation that can occur is an expansion that contains multiple irreducible factors. I.e. an expansion of the form

$$\frac{P(x)}{Q(x)} = \frac{a_1x+b_1}{(ax^2+bx+c)} + \frac{a_2x+b_2}{(ax^2+bx+c)^2} + \dots$$

For such an expansion, the factors can then be integrated either as logarithms or as powers through the substitution rule. I.e.

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1)$$
$$\int \frac{x}{(x^2+1)^2}dx = \frac{1}{2(x^2+1)}$$

**Exercise 60.** *Compute the following primitives:*

- $\int \frac{x}{x^2+x-2}$
- $\int \frac{1}{x^3+2x^2+x}$
- $\int \frac{2x}{(x+3)(3x+1)}dx$
- $\int \frac{x^4+1}{x^5+4x^3}dx$

**Exercise 61.** *Evaluate the following integrals:*

- $\int \frac{1}{(x+5)^2(x-1)}dx$
- $\int \frac{x}{x^2+4x+13}dx$
- $\int \frac{ds}{s^2(s-1)^2}$
- $\int \frac{4y^2-7y-12}{y(y+2)(y-3)}dy$
- $\int \frac{10}{(x-1)(x^2+y)}dy$

**Exercise 62.** *Start by using substitution to express the integrand as a rational function, then evaluate the integral using a partial fraction expansion*

- $\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}dx$
- $\int \frac{\sqrt{1+\sqrt{x}}}{x}dx$
- $\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx$
- $\int \frac{dx}{2\sqrt{x+3}+x}$

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