Analyse Numérique
Transformée de Fourier

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January 2022

Question 1 *Determine the Fourier series and display the approximation resulting from the first few coefficients for the function $f(t)$ defined as*

$$f(t) = \begin{cases} t & 0 \leq t \leq \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{\pi}{2} \leq t \leq \pi \\ \pi - \frac{t}{2} & \pi \leq t \leq 2\pi \end{cases}$$

Question 2 *Determine the Fourier series for $f(x) = H(x)$ the Heaviside unit step function, in the range $[-\pi, \pi]$, $f(x) = f(x + 2\pi)$. Deduce the value of the series*

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

Question 3 *Find the Fourier series of the function*

$$f(x) = \begin{cases} \sin(x/2) & 0 \leq x \leq \pi \\ -\sin(x/2) & \pi \leq x \leq 2\pi \end{cases}$$

with $f(x) = f(x + 2\pi)$

Question 4 *Show that the Fourier series for the function $f(x) = e^{ax}$, $-\pi < x < \pi$, (a is a real number) is given by*

$$\frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\left(a\cos(nx) - n\sin(nx)\right)\right\}$$

Deduce the value of the following four series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}, \quad \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}, \quad \sum_{n=1}^{\infty}\frac{1}{a^2+n^2}, \quad \sum_{n=-\infty}^{\infty}\frac{1}{a^2+n^2}$$

Question 5 *If the function f is defined as*

$$f(t) = \begin{cases} -t + e^t & -\pi \leq t < 0 \\ t + e^t & 0 \leq t < \pi \end{cases}$$

where $f(t) = f(t + 2\pi)$, sketch the graph of $f(t)$ for $-4\pi \leq t \leq 4\pi$ and obtain a Fourier series expansion for $f(t)$.

Question 6 *Find the Fourier series expansion of the function $f(t)$ where*

$$f(t) = \begin{cases} \pi^2 & -\pi < t < 0 \\ (t-\pi)^2 & 0 \leq t < \pi \end{cases}$$

and $f(t) = f(t + 2\pi)$. Deduce from this the value of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Question 7 *Find the Fourier series of the following functions, then use Julia to display the approximation of those functions following from the first few terms in the series*

- $\operatorname{sign}(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 0, & x = 0 \\ 1, & 0 < x \leq \pi \end{cases}$
- $|x|$, $-1 \leq x \leq 1$
- x, $-1 \leq x \leq 1$
- $f(x) = \begin{cases} 0 & -L \leq x \leq 0 \\ L & 0 < x \leq L \end{cases}$
- $f(x) = \sin x, \quad |x| \leq 2$

Question 8 *Show the following relations*

$$\int_{-L}^{L} e^{\frac{i\pi nx}{L}} e^{-\frac{i\pi mx}{L}}\, dx = \begin{cases} 0 & n \neq m \\ 2L & n = m \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = 0$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

Question 9 *Determine the Fourier transform of the function $f(t)$ defined by*

$$f(t) = \begin{cases} T + t & -T \leq t < 0 \\ T - t & 0 \leq t < T \\ 0 & \textit{otherwise} \end{cases}$$

Question 10 *Compute the Fourier transform of each of the following signals*

- $[e^{-at} \cos\omega_0 t]\, u(t)$, $a > 0$
- $e^{-3|t|} \sin 2t$
- $\left(\frac{\sin \pi t}{\pi t}\right)\left(\frac{\sin 2\pi t}{\pi t}\right)$