

## Wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L \quad t > 0 \\ u(0, t) = u(L, t) = 0 & t \geq 0 \end{cases}$$

$$u(x, 0) = g(x) \quad u_t(x, 0) = h(x) \quad 0 \leq x \leq L$$

→ Separation of variables

## Global Cauchy problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x, 0) = g(x) \quad u_t(x, 0) = h(x) \end{cases}$$

$$u(x, t) = \frac{1}{2} [g(x+ct) + g(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y) dy$$

d'Alembert

Today: Duhamel's Formula for the non homogeneous Cauchy problem.

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t) \\ u(x, 0) = 0 \quad u_t(x, 0) = 0 \quad x \in \mathbb{R} \end{cases}$$

Duhamel:  $s \geq 0 \quad w(x, t, s)$

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x, s, s) = 0 \quad w_t(x, s, s) = f(x, s) \end{cases} \quad x \in \mathbb{R}$$

→ using d'Alembert's formula

$$w(x, t, s) = \frac{1}{2c} \int_{x - c(t-s)}^{x + c(t-s)} f(y, s) dy$$

Now applying Duhamel's formula we get

$$u(x, t) = \int_0^t \omega(x, t; s) ds = \frac{1}{2c} \int_0^t \int_{x - c(t-s)}^{x + c(t-s)} f(y, s) dy ds$$

$$u_t = \int_0^t \omega_t(x, t, s) ds + \underbrace{\omega(x, t, t)}_{//}$$

$$u_{tt} = \underbrace{\omega_t(x, t, t)}_{//} + \int_0^t \omega_{tt}(x, t, s) ds$$

$$f(x, t) + \int_0^t \omega_{tt}(x, t, s) ds$$

$$u_{xx} = \int_0^t \omega_{xx}(x, t, s) ds$$

substituting  $u_{tt}$  and  $u_{xx}$  in the wave equation we  
get

$$u_{tt} - c^2 u_{xx} = f(x, t) + \int_0^t w_{tt}(x, t, s) ds - c^2 \int_0^t w_{xx}(x, t, s) ds$$

$= f(x, t)$  as  $w$  satisfies the wave equation

$$u(x, t) = \frac{1}{2} [g(x+ct) + g(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y) dy$$

$u(x, t) = F(x+ct) + G(x-ct)$  (\*)

to see this first let  $F(x+ct) = g(x+ct) + \frac{1}{2c} \int_0^{x+ct} h(y) dy$

 $G(x-ct) = g(x-ct) + \frac{1}{2c} \int_0^{x-ct} h(y) dy$

Another way to reach (\*) is through the  
canonical formulation

introduce  $\xi = \overbrace{x+ct}^{\gamma} \quad \eta = \overbrace{x-ct}^{\gamma}$

 $\Rightarrow x = (\xi + \eta)/2 \quad t = \frac{\xi - \eta}{2c}$

$$U(\xi, \gamma) = u\left(\frac{\xi + \gamma}{2}, \frac{\xi - \gamma}{2c}\right)$$

$$U_\xi = \underbrace{\frac{1}{2} u_x}_{\text{---}} + \frac{1}{2c} u_t$$

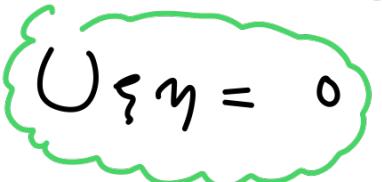
$$U_{\xi\gamma} = \frac{1}{2} u_{xx} \frac{dx}{d\gamma} + \frac{1}{2} u_{xt} \frac{dt}{d\gamma} + \frac{1}{2c} u_{tx} \frac{dx}{d\gamma} + \frac{1}{2c} u_{tt} \frac{dt}{d\gamma}$$

$$= \frac{1}{4} u_{xx} - \frac{1}{4c} u_{xt} + \frac{1}{4c} u_{tx} - \frac{1}{4c^2} u_{tt} = 0$$

$$u_{xx} - \frac{1}{c^2} u_t = 0$$

 follows from

Canonical Formulation

  $U_{\xi\gamma} = 0 \Rightarrow U = F(\xi) + G(\gamma)$

Substituting  $\xi = x + ct$      $\gamma = x - ct$  into an expression for  $U$

$$u(x, t) = F(x + ct) + G(x - ct)$$

## Classification of Second Order PDES

$$a u_{tt} + 2b u_{xt} + c u_{xx} + d u_t + e u_x + f u = g \quad *$$

Principal part

$$H(p, q) = ap^2 + 2bpq + cq^2 = 1 \quad *$$

When  $b^2 - ac > 0$  (\*) defines a hyperbola

$b^2 - ac = 0$  (\*) defines a parabola

$b^2 - ac < 0$  (\*) defines an ellipse

Diffusion / Heat equation  $\rightarrow u_{xx} - u_{tt} = 0$

$$C=2$$

parabolic

Laplace's equation  $\rightarrow C = 1 \quad u_{xx} + u_{tt} = 0$   
 $a = 1$  elliptic

Wave equation  $\rightarrow C = 1, a = -1$   
hyperbolic