Wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L \quad t > 0 \\ u(0,t) = u(L,t) = 0 & t \geq 0 \\ u(x,0) = g(x) \quad u_t(x,0) = h(x) & 0 \leq x \leq L \end{cases}$$

→ Separation of variables

Global Cauchy problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = g(x) \quad u_t(x,0) = h(x) \end{cases}$$

d'Alembert

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\, dy$$

Today: Duhamel's Formula for the non homogeneous Cauchy problem.

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = 0 \quad u_t(x,0) = 0 \quad x \in \mathbb{R} \end{cases}$$

Duhamel: $s \geq 0 \qquad w(x,t,s)$

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,s,s) = 0 \quad w_t(x,s,s) = f(x,s) \quad x \in \mathbb{R} \end{cases}$$

→ using d'Alembert's formula

$$w(x,t,s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy$$

Now applying Duhamel's formula we get

$$u(x,t) = \int_0^t w(x,t;s)\,ds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy$$

$$u_t = \int_0^t w_t(x,t,s)\,ds + \underbrace{w(x,t,t)}_{=0}$$

$$u_{tt} = \underbrace{w_t(x,t,t)}_{=f(x,t)} + \int_0^t w_{tt}(x,t,s)\,ds$$

$$= f(x,t) + \int_0^t w_{tt}(x,t,s)\,ds$$

$$u_{xx} = \int_0^t w_{xx}(x,t,s)\,ds$$

Substituting u_{tt} and u_{xx} in the wave equation we get

$$u_{tt} - c^2 u_{xx} = f(x,t) + \int_0^t w_{tt}(x,t,s)\,ds - c^2 \int_0^t w_{xx}(x,t,s)\,ds$$

$= f(x,t)$ as w satisfies the wave equation

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy$$

$$u(x,t) = F(x+ct) + G(x-ct) \quad (*)$$

to see this just let $F(x+ct) = g(x+ct) + \frac{1}{2c}\int_0^{x+ct} h(y)\,dy$

$$G(x-ct) = g(x-ct) + \frac{1}{2c}\int_{x-ct}^{0} h(y)\,dy$$

Another way to reach $(*)$ is through the canonical formulation

introduce $\xi = x+ct \qquad \eta = x-ct$

$$\Rightarrow \quad x = (\xi+\eta)/2 \qquad t = \frac{\xi-\eta}{2c}$$

$$U(\xi, \eta) = u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right)$$

$$U_\xi = \frac{1}{2} u_x + \frac{1}{2c} u_t$$

$$U_{\xi\eta} = \frac{1}{2} u_{xx} \frac{dx}{d\eta} + \frac{1}{2} u_{xt} \frac{dt}{d\eta} + \frac{1}{2c} u_{tx} \frac{dx}{d\eta} + \frac{1}{2c} u_{tt} \frac{dt}{d\eta}$$

$$= \frac{1}{4} u_{xx} - \frac{1}{4c} u_{xt} + \frac{1}{4c} u_{tx} - \frac{1}{4c^2} u_{tt} = 0$$

$$u_{xx} - \frac{1}{c^2} u_t = 0$$

follows from

Canonical Formulation

$$U_{\xi\eta} = 0 \Rightarrow U = F(\xi) + G(\eta)$$

Substituting $\xi = x + ct$ $\eta = x - ct$ into our expression for U

$$u(x,t) = F(x+ct) + G(x-ct)$$

Classification of Second Order PDES

$$a u_{tt} + 2b u_{xt} + c u_{xx} + d u_t + e u_x + h u = f \quad *$$

Principal part

$$H(p,q) = ap^2 + 2bpq + cq^2 = 1 \quad *$$

When $b^2 - ac > 0$ $(*)$ defines a hyperbola

$b^2 - ac = 0$ $(*)$ defines a parabola

$b^2 - ac < 0$ $(*)$ defines an ellipse

Diffusion / Heat equation $\longrightarrow$ $u_{xx} - u_t = 0$

$C = 1$

parabolic

Laplace's equation $\longrightarrow$ $c = 1$ $\quad u_{xx} + u_{tt} = 0$

$a = 1$

elliptic

Wave equation $\longrightarrow$ $C = 1, \; a = -1$

hyperbolic