Wave equation

→ Vibrating string ✓

→ Boundary condition ✓

→ group velocity / dispersion ✓

→ Conservation of energy ✓

→ Separation of variable ✓

→ (Well posedness + is the solution obtained by separation of variables well defined ?)

→ d'Alembert's formula

Recall the wave number k (= number of complete oscillations in the $[0, 2\pi]$ interval)

Wavelength $\lambda = \frac{2\pi}{k}$

angular frequency ω $\quad f = \frac{\omega}{2\pi}$

phase speed $c_p = \frac{\omega}{k} \longrightarrow$

Many oscillatory phenomena can be modelled by a superposition of harmonic waves with angular frequency that depends on k

if $\omega(k) = ck$ $\rightarrow$ all crests move with a constant speed.

if $\omega(k) \neq ck$ for some constant c, the crests move with a speed that depends on k

$$C_p = \frac{\omega(k)}{k}$$

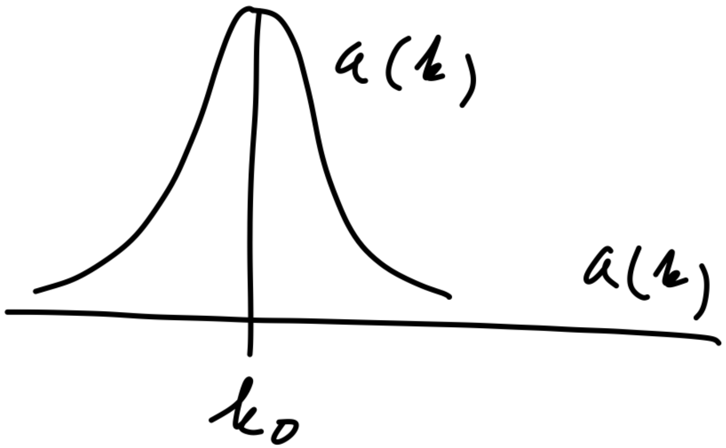
$\rightarrow$ the various components of the packet are going to separate or disperse

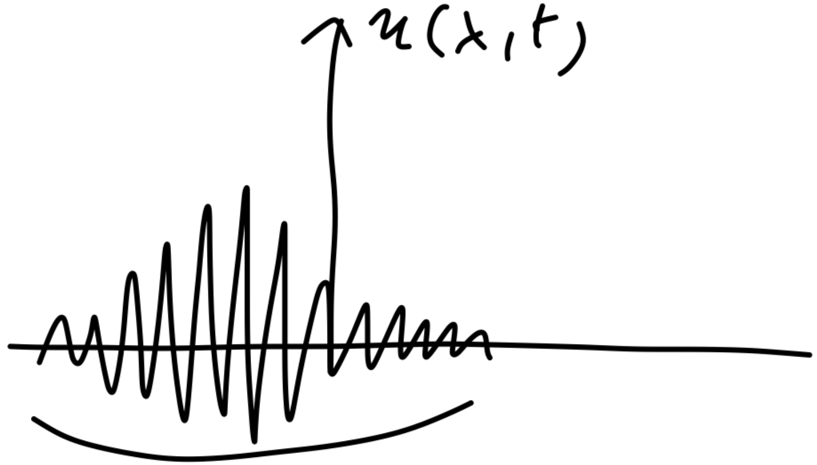
$\rightarrow$ In this case we let $C_g = \omega'(k)$ denote the group velocity

Wave packet = superposition of wave of different wave numbers k

$$u(x,t) = \int_{-\infty}^{\infty} a(k)\, e^{i(kx - \omega(k) t)}\, dk$$

$$a(k) = \exp(-8(k-14)^2)$$





Taking a Taylor expansion at k_0 for $\omega(k)$

$$\omega(k) \simeq \omega(k_0) + \omega'(k_0)(k-k_0)$$
$$= \omega(k_0) + c_g(k-k_0)$$

Substituting $(*)$ into $(*)$

$$u(x,t) = \underbrace{e^{i\{k_0 x - \omega(k_0)t\}}} \int_{k_0-\delta}^{k_0+\delta} \underbrace{a(k)\, e^{i(\overline{k-k_0})(x-\overline{c_g}t)}} dk$$

Conservation of energy

$$\boxed{E_{kin}(t) = \frac{1}{2}\int_0^L \rho_0(x)\, u_t^2 \, dx}$$

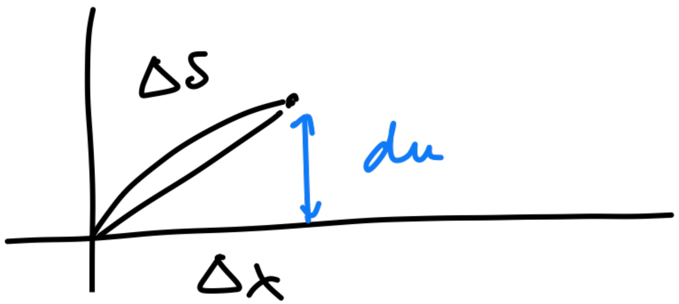
$\rho_0(x)$ = density

$u(x,t)$ = vertical displacement

$E_{pot}(t)$

$$dW = T_0 \cdot d = T_0(\Delta s - \Delta x)$$

where T_0 is the tension applied to the string



$$\Delta s \cong \sqrt{(\Delta x)^2 + (du)^2}$$

$$= dx\sqrt{1 + \left(\frac{du}{dx}\right)^2}$$

$$\int_0^L \tau_0 \left(\sqrt{1 + \left(\frac{du}{dx}\right)^2} - 1 \right) dx$$

$$\left(\sqrt{1 + \varepsilon^2} - 1 \right)$$

$$\approx 1 + \frac{\varepsilon^2}{2}$$

$$= \int_0^L \tau_0 \left(\frac{du}{dx}\right)^2 \cdot \frac{1}{2}\, dx = E_{pot}$$

$$E_{Tot} = E_{pot} + E_{kin}$$

$$= \int_0^L \tau_0 \left(\frac{du}{dx}\right)^2 \frac{1}{2}\, dx + \frac{1}{2} \int_0^L \rho_0(x)\, u_t^2\, dx$$

$$\frac{dE}{dt} = \int_0^L \tau_0 \, 2 \underbrace{\frac{du}{dx}}_{f} \underbrace{u_{xt}}_{g'} \, dx + \frac{1}{2} \int_0^L \rho_0(x) \, 2 u_t u_{tt} \, dx$$

$*$

integrating by parts the first term we get

$$* = \frac{1}{2} \left| u_x(x,t) u_t(x,t) \right|_0^L - \int_0^L \tau_0 \, 2 \, u_{xx} u_t \frac{1}{2} \, dx$$

$$\frac{dE}{dt} = \overbrace{\left| u_x(x,t) u_t(x,t) \right|_0^L} - 2 \int_0^L \tau_0 \, u_{xx} \, u_t \, \frac{1}{2} \, dx$$

$$+ \frac{1}{2} \int_0^L \rho_0(x) \, u_t u_{tt} \, dx$$

$$= \overbrace{u_x(L,t) u_t(L,t) - u_x(0,t) u_t(0,t)}$$

$$- \int_0^L u_t \left(- u_{xx} \tau_0 + \rho_0 u_{tt} \right) dx$$

$\Rightarrow$ if the end points do not move the energy is conserved.

We consider a general Cauchy problem

$$\begin{cases} u_{xx} - \frac{1}{c^2} u_{tt} = 0 \quad (*) \\ u(x,0) = g(x) \quad u_t(x,0) = h(x) \\ u(0,t) = 0 \quad u(L,t) = 0 \end{cases}$$

$\rightarrow$ As for the heat equation we let $u(x,t) = X(x)T(t)$

Substituting this in (*) we get

$$X''(x)\,T(t) - \frac{1}{c^2} X(x)\,T''(t) = 0$$

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = \lambda$$

#1 : $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = Ax + B$

#2 $\lambda > 0 \Rightarrow X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$

$$\left.\begin{array}{l} BC's \Rightarrow A + B = 0 \\ \Rightarrow Ae^{\sqrt{\lambda}L} + Be^{-\sqrt{\lambda}L} = 0 \end{array}\right\} \Rightarrow A = B = 0$$

#3 $\lambda < 0$ $X(x) = Ae^{\sqrt{-\lambda}ix} + Be^{-\sqrt{-\lambda}ix}$

BC's $\quad A+B=0$

$$A e^{\sqrt{-\lambda} i L} - A e^{-\sqrt{-\lambda} i L} = 0$$

$$\Rightarrow \quad 2i \sin \sqrt{-\lambda} L = 0$$

$$\sqrt{-\lambda} L = k\pi \quad \Rightarrow \quad \lambda = \left(\frac{k\pi}{L}\right)^2 \quad k = 1, \dots$$

$$X(x) = A_k \sin \frac{k\pi}{L} x$$

$$T''(t) = -\left(\frac{k\pi}{L}\right)^2 c^2 T(t) \quad \Rightarrow \quad T(t) = C_1 e^{i\left(\frac{k\pi}{L}\right) c t} + C_2 e^{-i\left(\frac{k\pi}{L}\right) c t}$$

$$U_k = \left[C_{1,k} e^{i\left(\frac{k\pi}{L}\right)ct} + C_{2,k} e^{-i\left(\frac{k\pi}{L}\right)ct} \right] \sin \frac{k\pi}{L} x \qquad (**)$$

→ k^{th} normal mode of vibration
k^{th} harmonic

→ it represents a standing wave with frequency $\frac{k\pi}{L}$

$$u(x,t) = B \cos kx \cos \omega t$$

→ the first harmonic and its frequency $\frac{1}{2L}$ are called fundamental harmonic and fundamental frequency if the initial conditions are of the form

$$u(x,0) = \underbrace{C \sin \mu_k x} \qquad u_t(x,0) = \underbrace{D \sin \mu_k x}$$

we get

$$(C_1 + C_2) = C$$

$$C_1 \, i \frac{k\pi}{L} C + C_2 \left(-i \frac{k\pi}{L}\right) C = D$$

$$\rightarrow \frac{k\pi}{L} i (C_1 - C_2) = D \qquad C_1 + C_2 = C$$

$$C_1 = \frac{1}{2}\left(C + \frac{DL}{k\pi i}\right) \qquad C_2 = \frac{1}{2}\left(C - \frac{DL}{k\pi i}\right)$$

Substituting this in (**)

$$U(x,t) = \left[\frac{1}{2}\left(C + \frac{DL}{k\pi i}\right) e^{i\frac{k\pi}{L}ct} + \frac{1}{2}\left(C - \frac{DL}{k\pi i}\right) e^{-i\frac{k\pi}{L}ct} \right] \sin \frac{k\pi}{L} x$$

$$\boxed{U(x,t) = \left[C \cos\left(\frac{k\pi}{L}ct\right) + \frac{DL}{k\pi} \sin \frac{k\pi}{L}ct \right] \sin \frac{k\pi}{L} x}$$

→ to get the general solution we sum the harmonics

$$U(x,t) = \sum_{k=1}^{\infty} \left[C_k \cos \frac{k\pi}{L} ct + \frac{D_k L}{k\pi} \sin \frac{k\pi}{L} ct \right] \sin \frac{k\pi}{L} x$$

If instead $h(x), g(x)$ are general functions,

$$U(x,0) = \sum_{k=1}^{\infty} A_k \sin \mu_k x$$

$$U_t(x,0) = \sum_{k=1}^{\infty} B_k \mu_k c \sin \mu_k x$$

with $g(x) = \sum_{k=1}^{\infty} \hat{g}_k \sin \mu_k x$

$$h(x) = \sum_{k=1}^{\infty} \hat{h}_k \sin \mu_k x$$

then set $\hat{h}_k = B_k \mu_k c$, $\hat{g}_k = A_k$

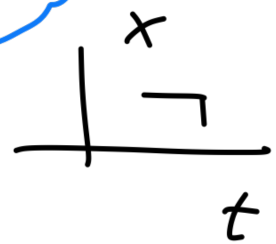
d'Alembert

let us go back to the Cauchy problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = g(x) \quad u_t(x,0) = h(x) \end{cases}$$

Note $\quad (\partial_t - c\,\partial_x)(\partial_t + c\,\partial_x)u = 0$

→ From this we can introduce $v = \partial_t u + c\,\partial_x u$



and we can look for the solution of

$$(\partial_t - c\,\partial_x)v = 0$$

which is a linear transport equation

$$v(x,t) = f(x+ct)$$

$$\frac{d}{dt} w(t) = f(x+ct)$$

in particular we thus have

$$\partial_t u + c\,\partial_x u = f(x+ct) \longrightarrow$$

$$\longrightarrow \boxed{u(x,t) = g(x-ct) + \int_0^t f(x-c(t-s)+cs)\,ds}$$

(***)

To fix $\psi(x)$ we use $u_t(x,0) = h(x)$

First note that

$$\int_0^t \psi(x - c(t-s) + cs)\, ds \qquad y = x - ct + 2cs$$

$$= \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y)\, dy \qquad \frac{dy}{ds} = 2c$$

let $F = \int \psi$

$$u_t(x,t) = g'(x-ct).(-c) + \frac{d}{dt}\left\{ \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y)\, dy \right\}$$

$$= -c\, g'(x-ct) + \frac{1}{2c} \frac{d}{dt}\left\{ F(x+ct) - F(x-ct) \right\}$$

$$u_t(x,t) = -c\,g'(x-ct) + \frac{1}{2c}\left\{\psi(x+ct)\,c + \psi(x-ct)(-c)\right\}$$

$$= -c\,g'(x-ct) + \frac{1}{2}\left\{\psi(x+ct) + \psi(x-ct)\right\}$$

taking $t=0$

$$u_t(x,0) = h(x) = -c\,g'(x) + \psi(x)$$

Together this thus gives

$$\psi(x) = h(x) + c\,g'(x)$$

Substituting this in (***) we get

$$u(x,t) = g(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y) + c\,g'(y)\,dy$$

$$u(x,t) = g(x-ct) + \frac{1}{2}g(x+ct) - \frac{1}{2}g(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy$$

$$\boxed{u(x,t) = \frac{1}{2}\left[g(x-ct) + g(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy}$$

Which is known as d'Alembert formula