Wave equation

deep water waves such as ocean waves

tidal waves (such as in tsunamis)

electromagnetic waves including radiowaves
X-rays
Microwaves

Gravitational waves

seismic waves / earthquakes

Physiological waves

Matter waves

- progressive or travelling waves

$$u(x,t) = g(x - ct)$$

$t=0 \to$ initial profile given by $g(x)$

as t increases the profile propagates with speed $|c|$

- Harmonic progressive waves

$$u(x,t) = A \exp(i(kx - \omega t))$$

$\to$ when considering the physics we will of course focus on the real (resp imaginary part)

$$A \cos(kx - \omega t)$$

In such waves we should list

- The wave amplitude $|A|$
- The wave number k which gives the number of oscillations in the interval $[0, 2\pi]$
- The wavelength $\lambda = \dfrac{2\pi}{k}$

(gives the distance between successive maximas (crests) or minimas (troughs))

- the angular frequency ω and the frequency

$$f = \frac{\omega}{2\pi}$$

(number of complete oscillations in one second [Hz] at fixed position)

- the wave or phase speed

$$c_p = \frac{\omega}{k}$$

which is the crest or trough speed

We call standing waves, waves of the form

$$u(x,t) = B \cos kx \cos \omega t$$

(can be obtained by superposing 2 harmonic waves with the same amplitude

$$A \cos(kx - \omega t) + A \cos(kx + \omega t) = 2A \cos kx \cos \omega t$$

• We call scalar plane wave, waves of the form

$$u(x,t) = f(\vec{k}\cdot\vec{x} - \omega t)$$

(disturbance propagating in the direction of $\vec{k}$ with speed $c_p = \frac{\omega}{|k|}$

The planes $\theta(x,t) = \vec{k}\cdot\vec{x} - \omega t = \text{cste}$ are known as wave fronts

• Finally Harmonic or monochromatic plane waves have the form

$$u(x,t) = A\exp(i(\vec{k}\cdot\vec{x} - \omega t))$$

$\vec{k}$ = wave number vector.

Derivation of the wave equation

We consider a simple classical model for the small transversal vibration of a tightly stretched horizontal string

We will make the following assumptions:

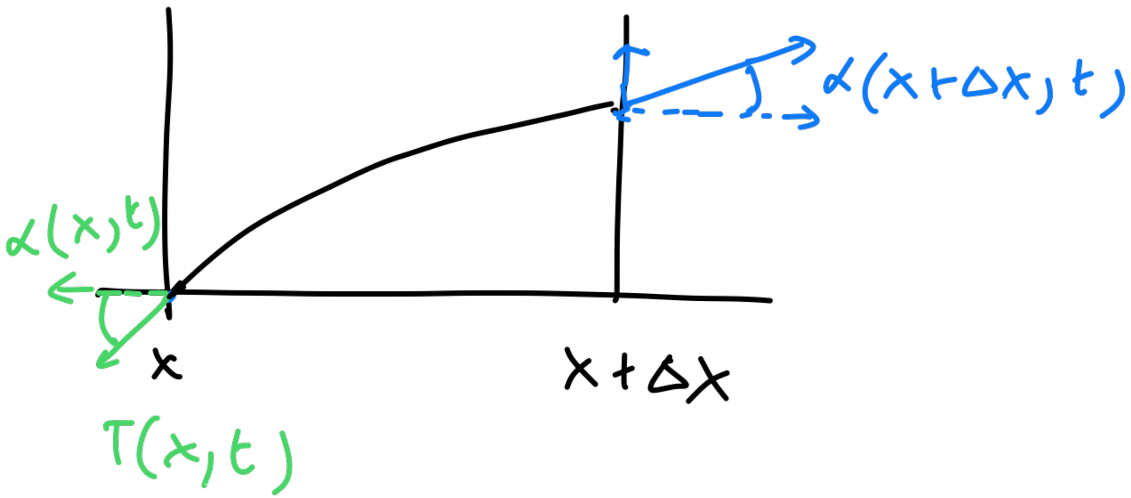
1. Vibrations of the string have small amplitude (changes in the slope of the string from horizontal equilibrium position are small)

2. each point x on the string only undergoes vertical displacement

$\boxed{3}$ the vertical displacement of a point depends on time and on its position on the string

$\boxed{4}$ String is perfectly flexible. No resistance to bending. Stress at any point given by a Tangential force $\vec{T}$ of magnitude T called tension

$\boxed{5}$ friction is negligible



density $\rho(x,t) = \rho_0(x)$

In order to apply Newton's law, let us list the forces

$$-T(x,t)\cos\alpha(x,t) + T(x+\Delta x, t)\cos\alpha(x+\Delta x, t) = 0$$

dividing by Δx and taking the limit we get

$$\frac{d}{dx}\Big(T(x,t)\cos\alpha(x,t)\Big) = 0$$

$$\Rightarrow \quad T(x,t)\cos\alpha(x,t) = T_0(t)$$

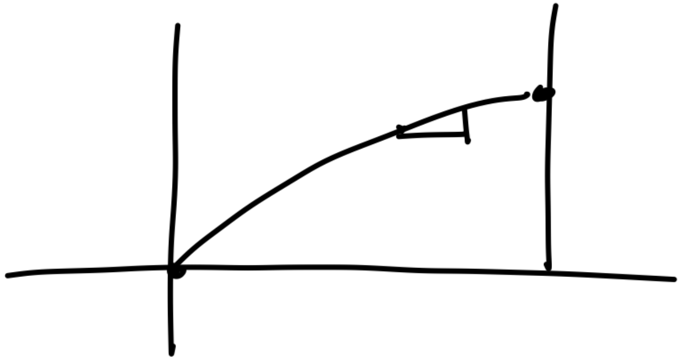
For the vertical forces we get

$$T(x+\Delta x,t)\sin\big(\alpha(x+\Delta x,t)\big) - T(x,t)\sin\big(\alpha(x,t)\big)$$

$$+\, Q(x,t)\,\rho_0(x)$$

From $\tau(x,t)\cos\alpha(x,t) = \tau_0(t)$

we have $\tau(x,t)\sin\alpha(x,t) = \tau_0(t)\cdot\frac{\sin\alpha(x,t)}{\cos\alpha(x,t)}$



$$\approx \tau_0(t)\,\frac{du}{dx}$$

where u = vertical displacement

$$\tau_0(t)\,u_x(x+\Delta x,t) - \tau_0(t)\,u_x(x,t) + Q(x,t)\,\rho_0(x)$$

Substituting in Newton's law, we get

$$\int_x^{x+\Delta x} \rho_0(x) \frac{\partial^2 u}{\partial t^2} \, dx = T_0(t)\, u_x(x+\Delta x, t) - T_0(t)\, u_x(x,t) \quad \int_x^{x+\Delta x} \rho_0(x)\, Q(x,t)\, dx$$

dividing by Δx and taking the limit

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} \rho_0(y)\, u_{tt}(y,t)\, dy = \lim_{\Delta x \to 0} T_0(t) \frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x}$$

$$+ \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} \rho_0(y)\, Q(y,t)\, dy$$

$$\Rightarrow \rho_0(x)\, u_{tt}(x,t) = \tau_0(t)\, u_{xx}(x,t) + \rho_0(x)\, Q(x,t)$$

$$\rightarrow \boxed{\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\tau_0(t)}{\rho_0(x)} \frac{\partial^2 u}{\partial x^2} + Q(x,t)}$$

$c^2(x,t)$

Which recovers the wave equation.

As for the heat equation we will focus on 3 main Boundary conditions:

<u>Dirichlet</u> describes the vertical displacement at the end points

$$u(0,t) = a(t) \qquad u(L,t) = b(t)$$

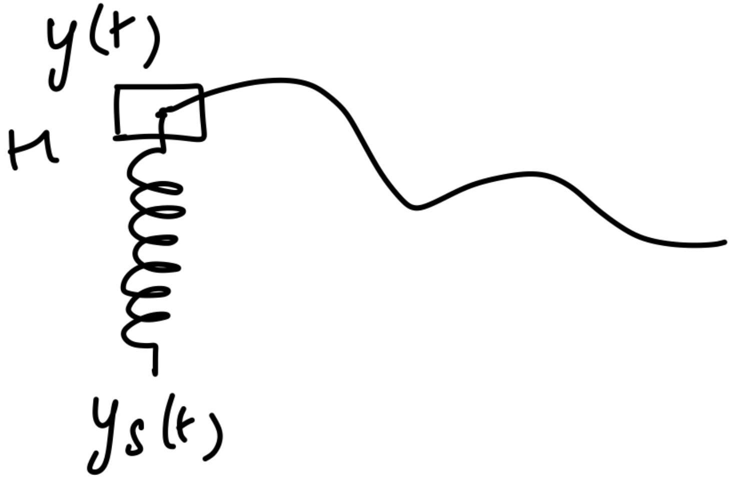
<u>Neumann data</u> describes an applied vertical tension at the end points

(equivalent to providing the slope of the string at the boundaries. Easiest illustration would be to set $\frac{\partial u}{\partial x} = 0$ which can be achieved by fixing the end points of the string to frictionless tracks

$$T_0 u_x(0,t) = a(t) \qquad T_0 u_x(L,t) = b(t)$$

Robin data arise when the ends of string are attached to a dynamical system

The most common setting is to suppose that the ends of the string are attached to a spring mass system



Since string is attached to the spring we have

$$u(0,t) = y(t)$$

vertical position of the mass satisfies an ODE following from Newton's law.

if spring has unstretched length ℓ and obeys Hooke's law with spring constant k

$$m \frac{d^2 y}{dt^2} = -k\big(y(t) - y_s(t) - \ell\big) + \underbrace{\text{other forces on the mass}}$$

includes tension from the string

$$\tau(0,t) \sin \alpha(0,t)$$

$$= \tau_0 \, \mathrm{tg}\, \alpha(0,t)$$

$$= \underbrace{\tau_0 \, u_x(0,t)}$$

$$m \frac{d^2}{dt^2} u(0,t) = -k\big(u(0,t) - y_s(t) - \ell\big) + \tau_0 \frac{\partial u}{\partial x}(0,t) + g$$

if we assume no acceleration and $g=0$

we recover

$$T_0 \frac{\partial u}{\partial x}(0,t) = k\left(u(0,t) - u_E\right) \quad \text{where } u_E = y_S - \ell$$

Neumann can be recovered by letting $k \to 0$