Today Method of characteristics

→ General conservation law

→ fan like characteristics

→ shock waves

↳ how to find the expression of the shock

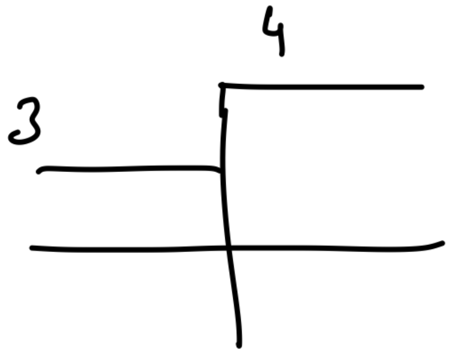
} 2 settings in which we will need extend our notion of solution

→ When can we certify existence of a classical solution

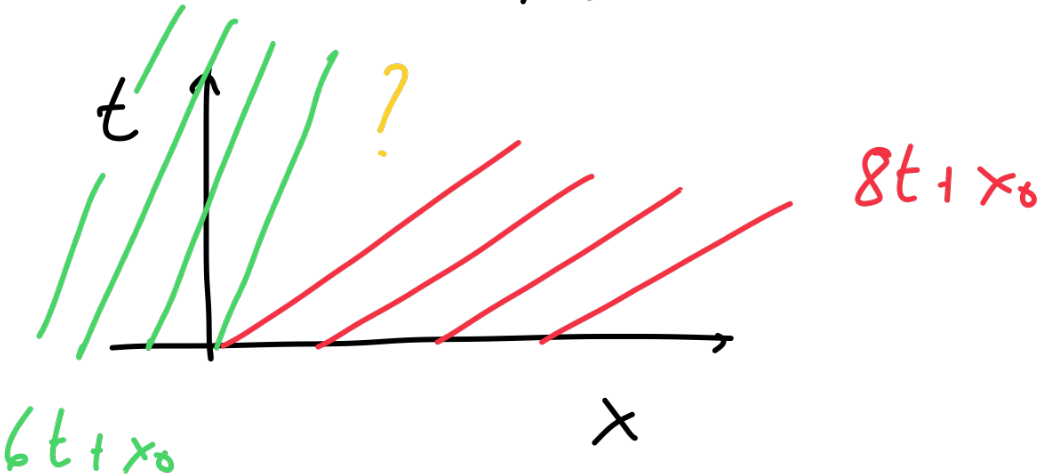
→ Weak solutions

I let us go back to our example

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$



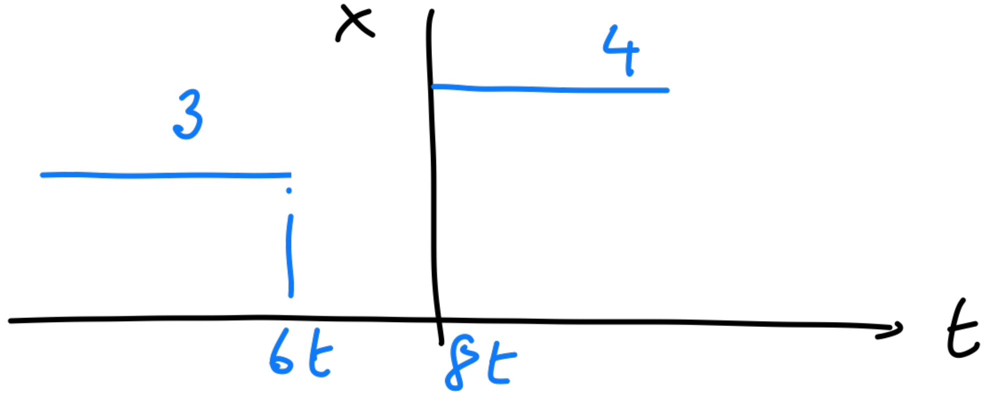

$$u(x,0) = \begin{cases} 3 & x < 0 \\ 4 & x > 0 \end{cases}$$

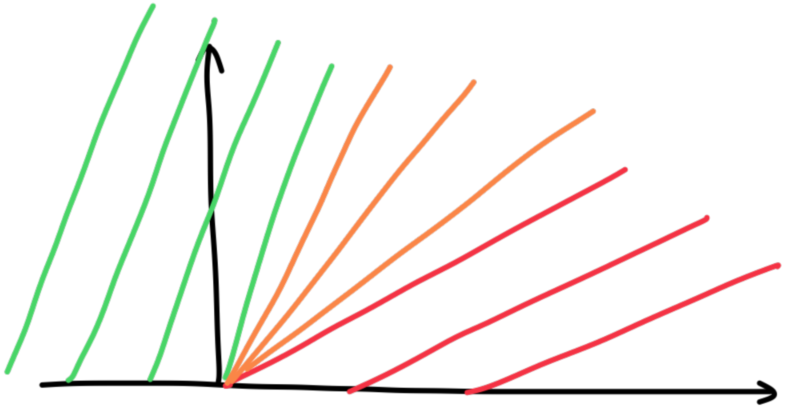


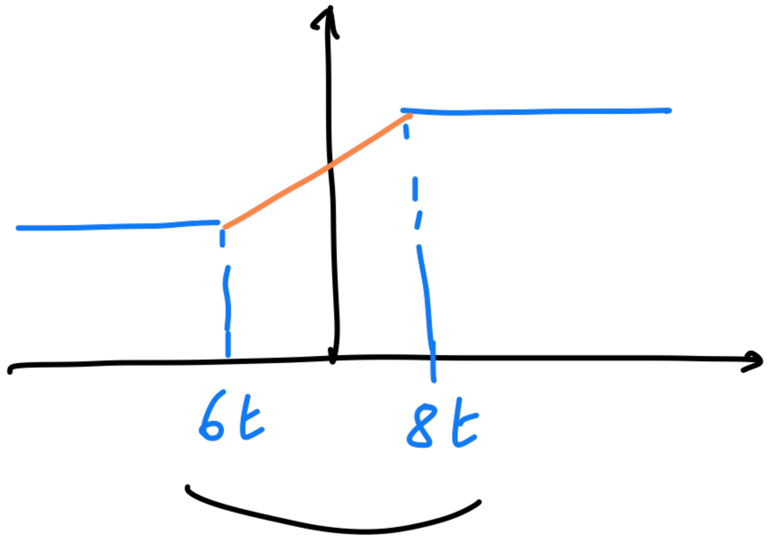

→ One fix: introduce fan-like characteristics

in this case the solution is defined as

$$u(x,t) = \begin{cases} 3 & x - 6t < 0 \\ 4 & x - 8t > 0 \end{cases}$$








$$\begin{cases} x(t) = 6t + x_0 & x_0 < 0 \\ x(t) = \varrho t & 6 < \varrho < 8 \\ x(t) = 8t + x_0 & x_0 > 0 \end{cases}$$

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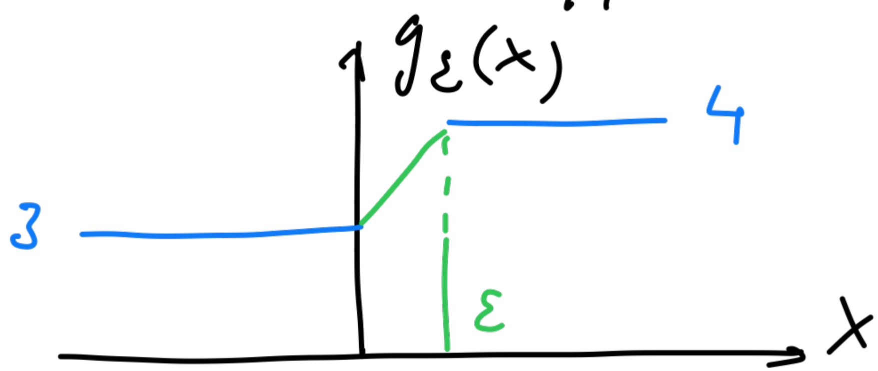
to get some more intuition: to understand the effect of the fan like characteristics, let us introduce the approximation

$$g_\varepsilon(x) = \begin{cases} 3 & x < 0 \\ 3 + x/\varepsilon & 0 \le x < \varepsilon \\ 4 & x \ge \varepsilon \end{cases}$$



then we can take $\lim_{\varepsilon \to 0}$ and determine if $\lim_{\varepsilon \to 0} u_\varepsilon$ is a valid solution

$$\frac{dx}{d\xi} = 2u \qquad \frac{dt}{d\xi} = 1 \qquad \frac{dz}{d\xi} = 0 \qquad \Gamma(s) = (s, 0)$$

$$\phi(s) = g_\varepsilon(s)$$

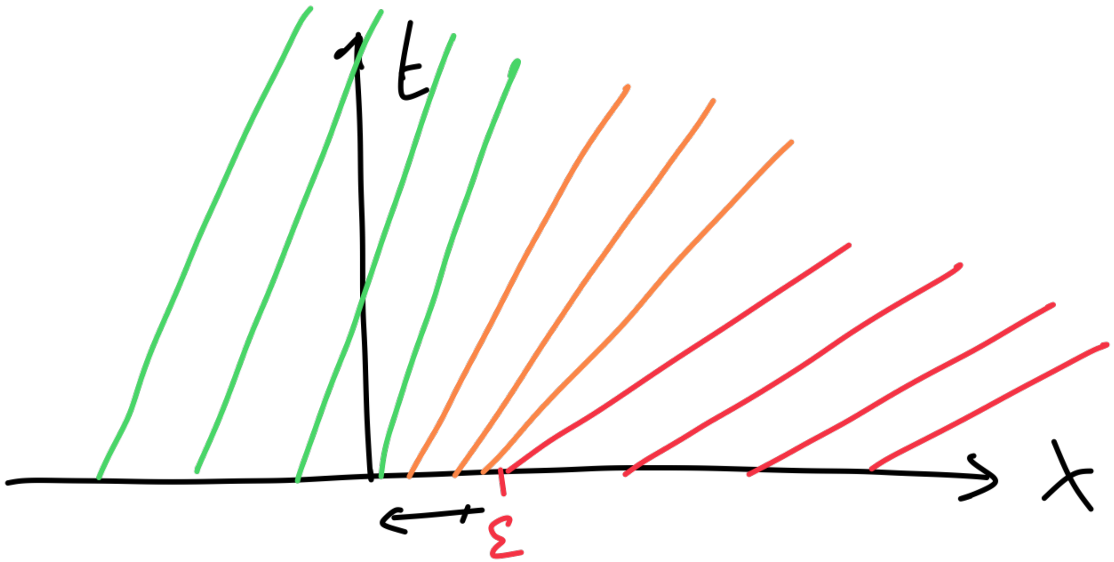
$$x = 2u\xi + c$$

$$= 2u\xi + s \qquad u(s,\xi) = g_\varepsilon(s)$$

$$x(t,s) = 2\, g_\varepsilon(s) \cdot t + s = 2 \begin{cases} 3 & s < 0 \\ 3 + s/\varepsilon & 0 \le s < \varepsilon \\ 4 & s > \varepsilon \end{cases} \cdot t + s$$

From here we see that the characteristics are defined

$$x(t) = \begin{cases} 6t + x_0 & x_0 < 0 \\ 6t + 2\dfrac{x_0 t}{\varepsilon} + x_0 & 0 \le x_0 < \varepsilon \\ 8t + x_0 & x_0 \ge \varepsilon \end{cases}$$



To get the solution we take our IC at the point on the characteristic

$$u(x,t) = g_\varepsilon(x_0)$$

where

$$x_0 = \begin{cases} x - 6t & \text{when } x_0 = x - 6t < 0 \\ (x-6t)\left(\frac{2t}{\varepsilon}+1\right)^{-1} & 0 \le (x-6t)\left(\frac{2t}{\varepsilon}+1\right)^{-1} < \varepsilon \\ x - 8t & x - 8t > \varepsilon \end{cases}$$

From this we get

$$u(x,t) = \begin{cases} 3 & x_0 < 0 \\ 3 + \dfrac{x_0}{\varepsilon} & 0 \le x_0 < \varepsilon \\ 4 & x_0 \ge \varepsilon \end{cases}$$

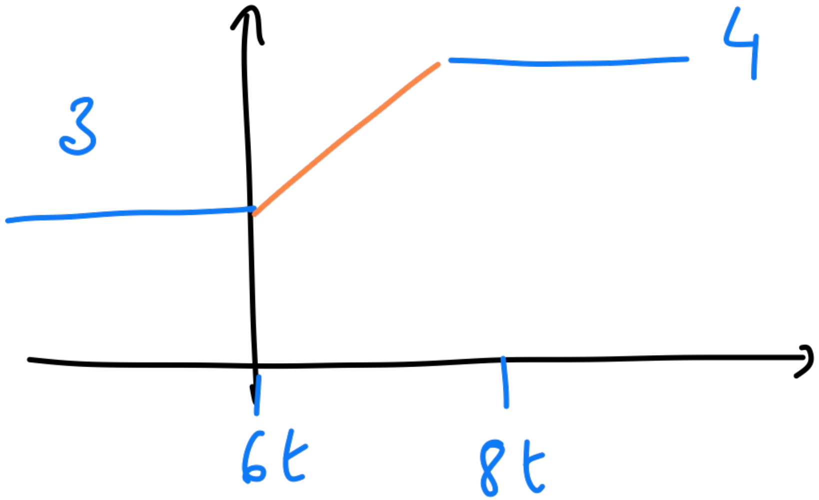
$$u(x,t) = \begin{cases} 3 & x - 6t < 0 \\ 3 + \dfrac{1}{\varepsilon}(x-6t)\left(\dfrac{2t}{\varepsilon} + 1\right)^{-1} & 0 \le (x-6t)\left(\dfrac{2t}{\varepsilon}+1\right)^{-2} < \varepsilon \\ 4 & x - 8t \ge \varepsilon \end{cases}$$

Which simplifies to

$$u_\varepsilon(x,t) = \begin{cases} 3 & x < 6t \\ 3 + \left(\dfrac{x-6t}{\varepsilon}\right)\left(\dfrac{\varepsilon}{2t+\varepsilon}\right) & x \geq 6t \text{ and } x < 8t+\varepsilon \\ 4 & x \geq 8t+\varepsilon \end{cases}$$

Now taking $\lim_{\varepsilon \to 0} u_\varepsilon(x,t)$ we recover

$$\lim_{\varepsilon \to 0} u_\varepsilon(x,t) = \begin{cases} 3 & x < 6t \\ 3 + \dfrac{x-6t}{2t} & 6t \leq x < 8t \\ 4 & x \geq 8t \end{cases}$$




→ Such a solution is known as a rarefaction or simple wave

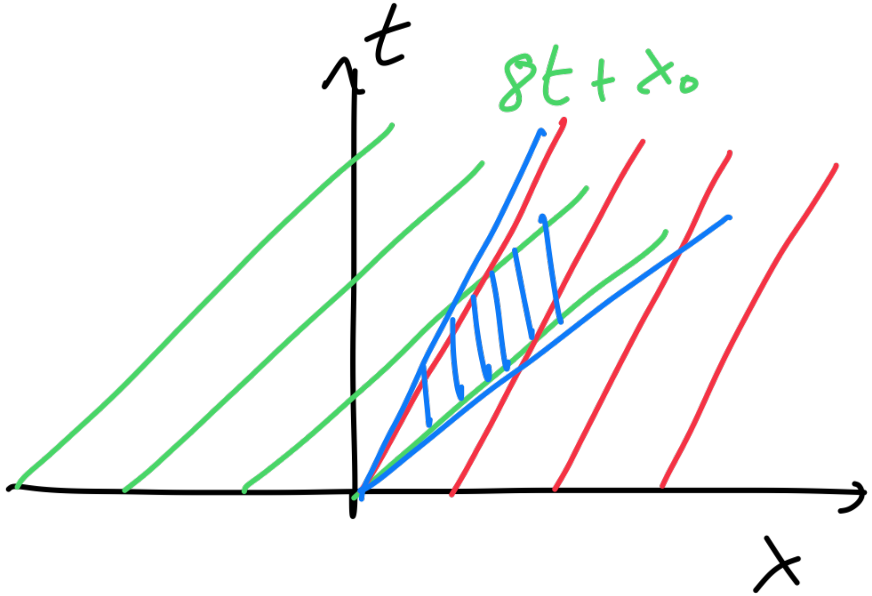
$$u(x,t) = g(x - vt)$$

## Ⅱ Shock waves

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

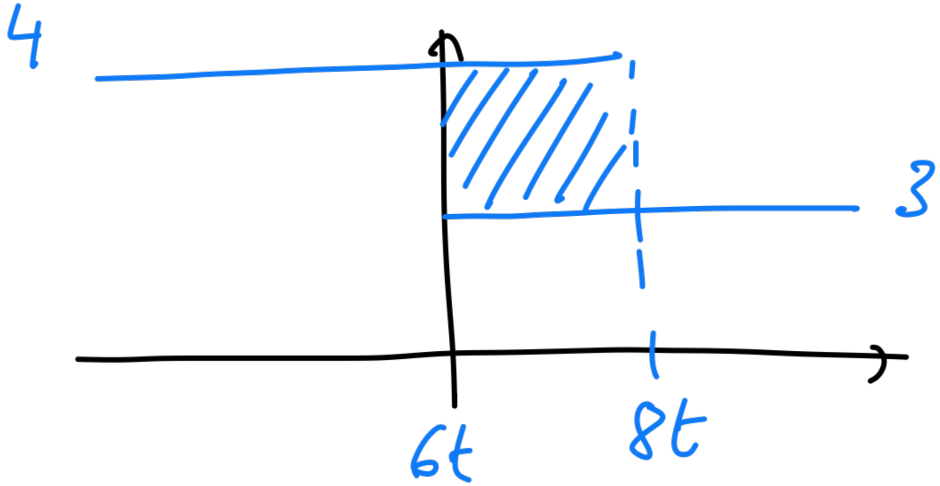
$$u(x,0) = \begin{cases} 4 & x<0 \\ 3 & x>0 \end{cases}$$

$$x(t) = \begin{cases} 8t + x_0 & x_0<0 \\ 6t + x_0 & x_0>0 \end{cases}$$




to get the solution, we trace $(x, t)$ all the way to the ICs.

$$u(x,t) = \begin{cases} 4 & x - 8t < 0 \\ 3 & x - 6t > 0 \end{cases}$$




→ How can we make sure that we get solutions that make sense? → general idea is to extend our solutions piecewise smooth function (or in the example above, functions smooth except on some parametric curve $\gamma(t) = x$)

General conservation law


$$\frac{\partial u}{\partial t} + q(u)_x = 0$$

$$\boxed{\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = q(u(x_1,t)) - q(u(x_2,t))}$$

$$\frac{d}{dt}\int_{x_1}^{\gamma(t)} u(x,t)\,dx + \frac{d}{dt}\int_{\gamma(t)}^{x_2} u(x,t)\,dx = q(u(x_1,t)) - q(u(x_2,t))$$

$$\frac{d}{dt}\int_{x_1}^{\gamma(t)} u(x,t)\,dx = \int_{x_1}^{\gamma(t)} u_t(x,t)\,dx + u(\gamma(t),t)\cdot\dot\gamma(t)$$

$$u\Big(\lim_{\substack{s\to\gamma(t)\\ <}} s,\ t\Big)\,\dot\gamma(t)$$

$$= \underbrace{u^-}_{\substack{\text{value of } u\\ \text{on the right of}\\ \gamma(t)}}(\gamma(t),t)\,\dot\gamma(t)$$

Doing the same with the $\int_{x_2}^{\gamma(t)}$ integral we get

$$\frac{d}{dt}\int_{\gamma(t)}^{x_2} u(x,t)\,dx = \int_{\gamma(t)}^{x_2} u_t(x,t)\,dx + u^+(\gamma(t),t)\,\dot\gamma(t)$$

Substituting this in the conservation law and taking

$\lim_{x_1 \nearrow \gamma(t)}$ and $\lim_{x_2 \searrow \gamma(t)}$ we recover

$$\dot{\gamma}(t)\, u^+(\gamma(t),t) - \dot{\gamma}\, u^-(\gamma(t),t) = q(u^+(\gamma(t),t)) - q(u^-(\gamma(t),t))$$

From this we get the equation that describes the shock

$$\dot{\gamma}(t) = \frac{q(u^+(\gamma(t),t)) - q(u^-(\gamma(t),t))}{u^+(\gamma(t),t) - u^-(\gamma(t),t)}$$

The condition on $\gamma(t)$ is known as the Rankine–Hugoniot condition