Recitation 6

Question 3

$$\begin{cases} a u_x + b u_y = 0 \\ u(0,y) = e^y \end{cases}$$

$u(\Gamma(s)) \Rightarrow \Gamma(s) = (0, s) \qquad z(s) = e^s$

$$\begin{cases} \dfrac{dx}{dt} = a & \dfrac{dy}{dt} = b & \dfrac{dz}{dt} = 0 \\ x(0) = 0 & y(0) = s & z(0) = e^s \end{cases}$$

$x = at \qquad y = bt + s \qquad z = e^s$

inverting for (s, t) we get $\quad t = \dfrac{x}{a}$

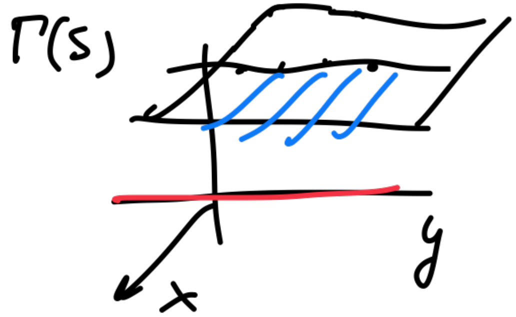
$$s = y - \frac{bx}{a}$$

From there we get the solution for u as

$$u(x,y) = z(s(x,y), t(x,y)) = e^{y - \frac{bx}{a}} .$$

Recitation 6

Question 5



$$u_x + y u_y = y^2 \qquad u(0,y) = \sin y$$

$$a(x,y,u)\frac{du}{dx} + b(x,y,u)\frac{du}{dy} = c(x,y,u)$$

$$\frac{dx}{dt} = a(x,y,u) \qquad \frac{dy}{dt} = b(x,y,u) \qquad \frac{dz}{dt} = c(x,y,u)$$

$$u(\Gamma(s)) = \phi(s)$$

in this case $\Gamma(s) = (0,s) \quad \phi(s) = \sin s$

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = y \qquad \frac{dz}{dt} = y^2$$

$$x(0) = 0 \qquad y(0) = s \qquad z(0) = \sin s$$

$$\underbrace{x(t,s) = t}$$

$$\frac{dy}{dt} \cdot \frac{1}{y} = 1$$

$$\underbrace{y(t,s) = s e^{t}}$$

$$\frac{dz}{dt} = s^2 e^{2t}$$

$$z(t,s) = \frac{s^2 e^{2t}}{2} + C$$

$$z(t,s) = \frac{s^2 e^{2t}}{2} + \sin(s) - \frac{s^2}{2}$$

taking $t = x$, $s = ye^{-t} = ye^{-x}$ we get

$$u(x,y) = z(t(x,y), s(x,y))$$

$$u(x,y) = y^2 \frac{e^{-2x} e^{2x}}{2} + \sin(ye^{-x}) - \frac{y^2 e^{-2x}}{2}$$

$$u(x,y) = \frac{y^2}{2} + \sin(ye^{-x}) - \frac{y^2 e^{-2x}}{2}$$

Question 6

$$u_x + y u_y = u^2 \qquad u(0,y) = \sin y \qquad \Gamma(s) = (0, s)$$

$$\phi(s) = \sin(s) = u(\Gamma(s))$$

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = y \qquad \frac{dz}{dt} = u^2 = \frac{du}{dt}$$

$$x(0) = 0 \qquad y(0) = s \qquad z(0) = \sin(s)$$

$x(t) = t \quad y(t) = s e^t$

$$\frac{du}{dt} \cdot \frac{1}{u^2} = 1$$

$$\int \frac{du}{dt'} \frac{1}{u(t')^2} dt' = \int 1 \, dt'$$

$$-\frac{1}{u(t)} = t + c$$

$$\Rightarrow \quad u(t) = \frac{-1}{t+c}$$

$$u(s,t) = \frac{-1}{t + \frac{1}{\sin(s)}} = \frac{-\sin(s)}{\sin(s)t+1}$$

using

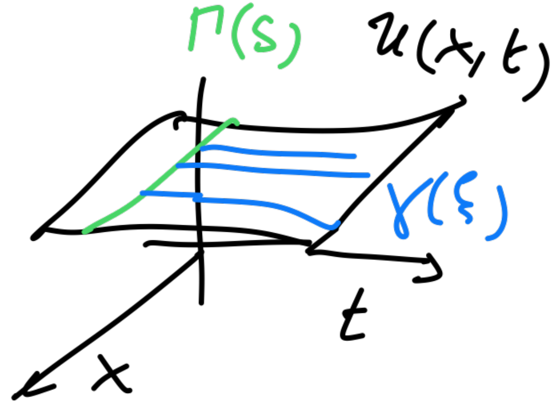
$(t,s) = (x, y e^{-x})$

we get

$$u(x,y) = \frac{-\sin(ye^{-x})}{\sin(ye^{-x})x + 1}$$

Question 7 We look for the solution of Burgers equation

$$u_t + u u_x = -x \qquad u(x,0) = \phi(x)$$



$$\Gamma(s) = (s, 0) \qquad u(\Gamma(s)) = \phi(s)$$

$$\frac{dt}{d\xi} = 1 \qquad \frac{dx}{d\xi} = u \qquad \frac{dz}{d\xi} = -x$$

$$t(\xi) = \xi + c$$

$$t(0) = 0 \qquad x(0) = s \qquad z(0) = \phi(s)$$

$$\Rightarrow \quad t(\xi) = \xi \qquad \begin{cases} \dot{x} = z \\ \dot{z} = -x \end{cases}$$

differentiating one more time, we get

$$\begin{cases} \ddot{x} = -x \\ \ddot{z} = -z \end{cases} \qquad x(0) = s \qquad \dot{x}(0) = \phi(s)$$

$$x(\xi) = A\cos\xi + B\sin\xi$$

$$= s\cos\xi + \phi(s)\sin\xi \longleftarrow \text{using } \begin{array}{l} x(0) = s \\ \dot{x}(0) = \phi(s) \end{array}$$

For $z(\xi)$ taking $Z(\xi) = \dot{x}(\xi) = -s\sin\xi + \phi(s)\cos\xi$

using $\xi = t$

as well as

$$\begin{cases} x = \left(s \cos\xi + \phi(s)\sin\xi\right) \cos\xi \\ \left(z = \left(\phi(s)\cos\xi - s\sin\xi\right)\right) \sin\xi \end{cases}$$

$$x\cos\xi - z\sin\xi = s\cos^2\xi + s\sin^2\xi$$

$$\boxed{s = x\cos\xi - z\sin\xi}$$

$$u(x,t) = z\left(s(x,t), \xi(x,t)\right)$$

$$= \phi(x\cos t - u\sin t)\cos t - (x\cos t - u\sin t)\sin t$$

Question 9

$$(y+u)u_x + y\,u_y = x-y \qquad u(x,1) = 1+x$$

$$\Gamma(s) = (s,1) \qquad x(s) = s \qquad y(s) = 1 \qquad u(\Gamma(s)) = 1+s$$

$$\frac{dx}{dt} = y+u \qquad \frac{dy}{dt} = y \qquad \frac{dz}{dt} = x-y$$

$$x(0) = s \qquad y(0) = 1 \qquad z(0) = 1+s$$

$$y(t) = e^t \qquad \begin{cases} \dot{x} = y+u \\ \dot{u} = x-y \end{cases}$$

$$\rightarrow t = \log y$$

Approach #1 : differentiate both side and solve the second order ODE

Approach #2 set $v = x + u$

$w = x - u$

$$\dot{w} = \frac{dx}{dt} - \frac{du}{dt} = y + u - x + y$$
$$= 2y - w$$

$$\dot{w} + w = 2e^{t} \qquad w(0) = -1$$

$$\rightarrow \quad w(t) = Ae^{-t} + e^{t} \xrightarrow{\ \downarrow \text{using}\ } \quad A + 1 = -1$$
$$A = -2$$

$$\dot{v}(t) = \frac{dx}{dt} + \frac{du}{dt} = y + u + x - y$$

$$= u + x = v$$

$$v(0) = x(0) + u(0) = 2s + 1$$

$$v(t) = Ae^{-t}$$

$t = \log y$

$$v(t,s) = (2s+1)e^{-t}$$

$$\Rightarrow \begin{cases} x + u = (2s+1)e^{-t} \\ x - u = -2e^{-t} + e^{t} \end{cases} \longrightarrow u = x - e^{t} + 2e^{-t} \quad *$$

$$x = \frac{1}{2}\left((2s+1)e^{-t} + e^{t} - 2e^{-t}\right)$$

inverting for s we get

$$\begin{cases} s = \left[(2x - e^t + 2e^{-t}) e^t - 1 \right]^{1/2} \\ t = \log y \end{cases} \Rightarrow \text{substitute in } (*)$$