First order Equations

→ linear transport equation

→ conservation laws

$$u_t + (q(u))_x = 0$$

→ quasilinear First order equations

→ Method the characteristics

Today → existence and uniqueness in the classical setting

→ weak solutions (solutions that are not smooth)

→ What conditions do weak solutions have to satisfy

→ Uniqueness → How do we select among multiple weak solutions! (Entropy)

(I) Existence and uniqueness of classical solutions

$$\begin{cases} u_t + q(u)_x = 0 \\ u(x,0) = g(x) \end{cases}$$

along our characteristic $x(t)$, the solution will obey

(i) $u(x(t),t) = g(x_0)$

(ii) $\frac{d}{dt} u(x(t),t) = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dt} = \underline{u_x \dot{x}(t) + u_t = 0}$ (*)

From the conservation law we also have

$$u_t + q'(u) u_x = 0$$

along the characteristic we therefore recover

$$\rightarrow \quad u_t(x(t),t) + q'(\underbrace{u(x(t),t)}_{g(x_0)}) u_x(x(t),t) = 0 \quad (*)$$

Subtracting (*) from (*) we get

$$u_x \, \dot{x} \qquad - q'(u(x(t),t)) \, u_x(x(t),t) = 0$$

Now assuming $u_x \neq 0$, we end up with

$$\dot{x}(t) = q'(g(x_0))$$

From this simple ODE we recover the characteristic

$$x(t) = q'(g(x_0)) t + x_0$$

From there our solution for $u(x,t)$ is given by

$$u(x,t) = g(x - q'(g(x_0)) t)$$

Theorem [see Theorem 6.1.1 in Dafermos]

We consider the homogeneous scalar conservation law

$$\begin{cases} \partial_t u(x,t) + q(u)_x = 0 \\ u(x,0) = g(x) \end{cases} \qquad (*)$$

assume that $g(x)$ is bounded and Lipschitz continuous

if we further assume $\frac{d}{dy} q'(g(y)) \geq 0$

then there exists a classical solution u of $(*)$

on $[0,\infty]$. Furthermore if $g(x)$ is C^k, so is u

the $\frac{d}{dy} q'(g(y))$ follows from the implicit function theorem and corresponds to requiring the characteristics to have increasing slopes.

→ the result applies to classical solutions ($u \in C^2([0,\infty) \times \mathbb{R}_+)$)

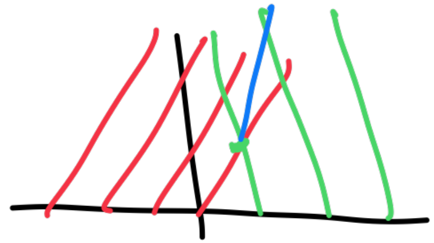
→ the proof of the theorem relies on the IMPLICIT Function theorem

$$1 + t\, \overbrace{q''(g(x_0))\, g'(x_0)}\, \underbrace{}_{\frac{d}{dy} q'(g(y))} \geq 0$$

which requires the more general condition

this more general condition can be used to determine the time at which we lose the smoothness of the solution (i.e corresponding to the first appearance of the shock)

$$t = \frac{-1}{q''(g(x_0))\, g'(x_0)}$$



in order to find the first time, we look for

$$\tilde{t} = \min_{x_0} \frac{-1}{q''(g(x_0))\, g'(x_0)}$$

Moreover, substituting this in the equation of the characteristic we can recover the associated position $\tilde{x}$

$$\tilde{x} = q'(g(\tilde{x}_0)) \cdot \tilde{t} + \tilde{x}_0$$

where $\tilde{x}_0 = \operatorname{argmin}_{x_0} \frac{-1}{q''(g(x_0))\, g'(x_0)}$

In order to account for non smooth solutions, we will consider a test function v which itself is smooth and has compact support

Multiplying our PDE with v and integrating

$$\int_0^\infty \int_{\mathbb{R}} [u_t + q(u)_x] v \, dx \, dt = 0$$

using integration by parts, we get

$$\int_0^\infty \int_{\mathbb{R}} u_t v \, dx \, dt = -u(x,0) v(x,0) - \int_0^\infty \int_{\mathbb{R}} u v_t \, dx \, dt$$

$$\int_0^\infty \int_{\mathbb{R}} q(u)_x v \, dx\, dt = -\int_0^\infty \int_{\mathbb{R}} q(u)\, v_x \, dx\, dt \qquad \int (u(x,0) - g(x))\, v$$

$$\int_0^\infty \int_{\mathbb{R}} [u v_t + q(u) v_x]\, dt\, dx + \int_{\mathbb{R}} \overbrace{u(x,0)}^{g(x)} v(x,0) = 0 \qquad (**)$$

is known as the weak or integral formulation.

1) How does (**) relate to classical solutions?

① → if u satisfies (**) for every test function v, we can always take a test function v vanishing at 0 which gives

$$\int_0^\infty \int_{\mathbb{R}} [u v_t + q(u) v_x] \, dx \, dt$$

$$= -\int_0^\infty \int_{\mathbb{R}} u_t v \, dx \, dt - \int_0^\infty \int_{\mathbb{R}} q(u)_x v \, dx \, dt$$

for every test function $v(x,t)$ with $v(x,0)=0$

$$\Rightarrow \quad u_t - q(u)_x = 0 \quad \forall \, (x,t) \in \mathbb{R} \times (0,\infty)$$

(**)

② Conversely we can choose $v(x,t)$ that do not vanish at 0. In this case using (**), the integral formulation (**) reduces to

$$\int_{\mathbb{R}}\int_0^\infty \left[\underbrace{u v_t + q(u) v_x}\right] + \int_{\mathbb{R}} v(x,0)\, u(x,0) = 0$$

$$= -\int_{\mathbb{R}}\int_0^\infty u_t v - \int_{\mathbb{R}} u(x,0)\, v(x,0) - \int_{\mathbb{R}}\int_0^\infty (q(u))_x v$$

$$+ \int_{\mathbb{R}} v(x,0)\, g(x)$$

$$\Longrightarrow \int_{\mathbb{R}} (u(x,0) - g(x))\, v(x,0) \quad \forall v$$

Which $u(x,0) - g(x) = 0 \quad \forall x$

All in all the previous reasoning leads to the following result:

A function $u \in C^2(\mathbb{R} \times [0, +\infty])$ is a classical solution of the conservation law if and only if it satisfies the weak / integral formulation for every test function v.

We are now ready to extend our notion of solution

<u>Def</u> : A function u on $\mathbb{R} \times [0, +\infty)$ is called a weak solution of the conservation law if the integral formulation holds for every test function v

<u>theorem</u> let $u : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ be a piecewise C^2 (smooth) function then u is a weak solution of the conservation law if and only if

1) u is a classical solution in the region where it is C^2

2) the Rankine Hugoniot conditions hold along each discontinuity line $s_j(t)$

$$\dot{s}_j(t) = \frac{q(u_+(s_j(t),t)) - q(u_-(s_j(t),t))}{u_+(s_j(t),t) - u_-(s_j(t),t)}$$

What can we say regarding the uniqueness of weak solutions?

To discuss uniqueness, let us consider Burgers

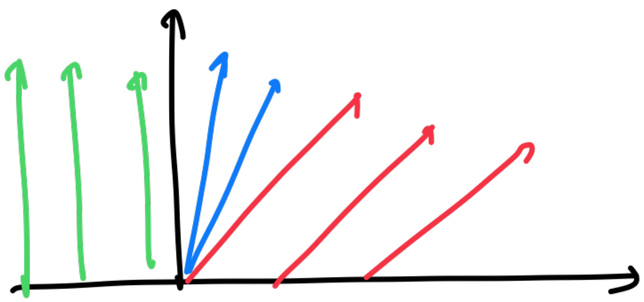
$$u_t + u u_x = u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$\frac{dt}{d\xi} = 1 \qquad t(0) = 0$$

$$\frac{dx}{d\xi} = u \qquad x(0) = s$$

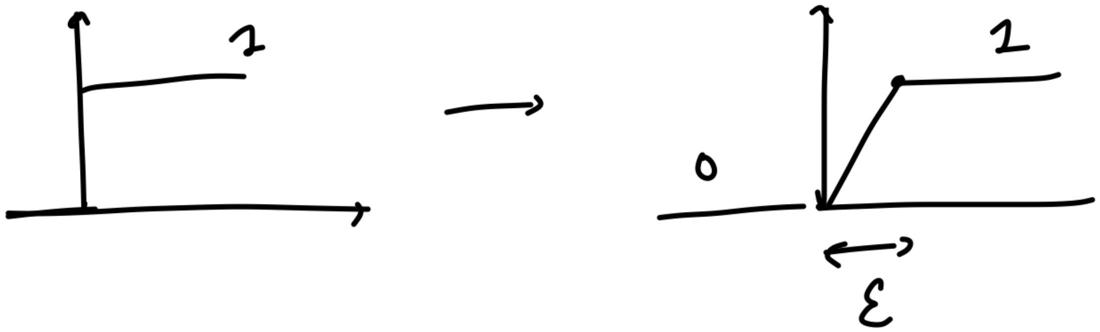
$$\frac{dz}{ds} = 0 \qquad z(s) = g(s)$$



$$\rightarrow \quad x = g(s) \cdot \xi + s$$

$$= g(s)\, t + s$$

$$x(t) = \begin{cases} s & s < 0 \\ t + s & s > 0 \end{cases}$$



For the rarefaction wave we get

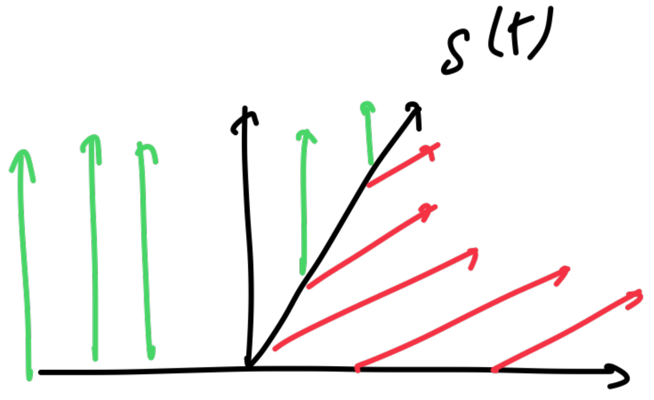
$$u(x,t) = \begin{cases} 0 & x \leq 0 \\ x/t & 0 < x < t \\ 1 & x \geq t \end{cases}$$

$$\dot{s}(t) = \frac{q(u_+) - q(u_-)}{u_+ - u_-} = \frac{1/2}{1} = 1/2$$

$$1 + t\, q''(g(x_0))\, g'(x_0)$$

$$s(t) = 1/2\, t$$

$$t = \frac{-1}{q''(\underset{1}{\cancel{g(x_0)}})\, g'(x_0)}$$



→ When there are multiple weak solutions, we want to select the solution that corresponds to the physics.

→ Generally speaking we want our solution to satisfy the laws of thermodynamics

Concretely this translates into the fact that the physical solution is the one that satisfies

$$u_+(x,t) < u_-(x,t) \qquad q'' > 0$$

$$u_-(x,t) < u_+(x,t) \qquad q'' < 0$$

In the case of a shock, these conditions translate into

$$q'(u_+(s,t)) < \dot{s}(t) < q'(u_-(x,t)) \qquad q'' > 0$$

$$q'(u_-(s,t)) < \dot{s}(t) < q'(u_+(x,t)) \qquad q'' < 0$$