MATH-UA 9263 - Partial Differential Equations Recitation 7: Conservation laws (part II), shock waves and rarefaction waves

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Question 1 (Salsa) *Solve the Burgers equation $u_t + uu_x = 0$ with initial data*

$$g(x) = \begin{cases} 1 & x \leq 0 \\ 1 - x & 0 < x < 1 \\ 0 & X \geq 1 \end{cases}$$

Question 2 *We consider the conservation law*

$$\begin{cases} u_t + (q(u))_x = 0 \\ u(x, 0) = g(x) \end{cases}$$

with $q(u) = \frac{u(2-u)}{2}$ *as well as*

$$g(x) = \begin{cases} 0 & x < 1 \\ 1 & x \in (1, 2) \\ 2 & x > 2 \end{cases}$$

Question 3 *We consider the conservation law*

$$\begin{cases} u_t + (q(u))_x = 0 \\ u(x, 0) = g(x) \end{cases}$$

with $q(u) = \frac{u(2-u)}{2}$ *as well as*

$$g(x) = \begin{cases} 2 & x < 1 \\ 0 & x \in (1, 2) \\ 1 & x > 2 \end{cases}$$

Question 4 *Solve the following equations subject to the initial condition $\rho(x, 0) = f(x)$*

i) $u_t + cu_x = e^{-3x}$

ii) $u_t - t^2 u_x = -u$

iii) $u_t + xu_x = t$

Question 5 *Solve the following problems*

i) $u_t + 3uu_x = 0, \qquad u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & 0 < x < 1 \\ 4 & x > 1 \end{cases}$

ii) $u_t + 6uu_x = 0,$ *for* $x > 0$ *only,* $\begin{cases} u(x,0) = 5 & x > 0 \\ u(0,t) = 2 & t > 0 \end{cases}$

Question 6 *Solve the following problems (including shocks and rarefaction waves). if a non uniform shock occurs, only provide the differential equation*

a) $u_t + u^2 u_x = 0, \quad u(x,0) = \begin{cases} -1 & x < 0 \\ 2 & x > 0 \end{cases}$

b) $u_t + u^2 u_x = 0, \quad u(x,0) = \begin{cases} 2 & x < 0 \\ -1 & x > 0 \end{cases}$

Question 7 (Non uniqueness of shock velocity) *The shock velocity* $\dot{s}(t) = \frac{q(u_L) - q(u_R)}{u_L - u_R}$ *for the equation*

$$u_t + (q(u))_x = 0$$

is not unique because mathematically the PDE can be multiplied by any function of u *and the resulting shock velocity would be different. As an example, if we consider a shock for* $u_R = 4$ *and* $u_L = 3$ *with* $q(u) = u^2$*, the shock velocity gives* $\dot{s} = 7$*. if we multiply the PDE by* u *however, we get the equation* $(u^2/2)_t + (2u^3/3)_x = 0$ *(which can be understood as a conservation law for* u^2*). In this case, the shock velocity is given by* $\dot{s} = 148/21$*. Solve the following problems first assuming that* ρ *is conserved, then assuming that* ρ^2 *is conserved*

a) $u_t + u^2 u_x = 0, \quad u(x,0) = \begin{cases} 4 & x < 0 \\ 3 & x > 0 \end{cases}$

b) $u_t + 4uu_x, \quad u(x,0) = \begin{cases} 3 & x < 1 \\ 2 & x > 1 \end{cases}$

c) $u_t + 3uu_x = 0 \quad u(x,0) = \begin{cases} 4 & x < 0 \\ 2 & 0 < x < 1 \\ 1 & x > 1 \end{cases}$

Question 8 (Salsa) *Show that, for every* $\alpha > 1$*, the function*

$$u_\alpha(x,t) = \begin{cases} -1 & 2x \leq -(1+\alpha)t \\ -\alpha & -(1+\alpha)t < 2x < 0 \\ \alpha & 0 < 2x < (\alpha+1)t \\ 1 & (\alpha+1)t \leq 2x \end{cases}$$

is a weak solution of the problem

$$\begin{cases} u_t + uu_x = 0, & t > 0, x \in \mathbb{R} \\ u(x,0) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases} \end{cases}$$

Question 9 (Salsa) *Let* $D = \left\{(x,y)|y > x^2\right\}$ *and* $a = a(x,y)$ *be a continuous function in* $\overline{D}$.

1) *Given* $g \in C(\mathbb{R})$, *check the solvability of the linear problem*

$$\begin{cases} a(x,y)u_x - u_y = -u & (x,y) \in D \\ u(x,x^2) = g(x) & x \in \mathbb{R} \end{cases}$$

2) *Examine the cases* $a(x,y) = y/2$ *and* $g(x) = \exp\left(-\gamma x^2\right)$ *where* γ *is a real parameter.*

Question 10 (Salsa) *Solve the Cauchy problem*

$$\begin{cases} xu_x - yu_y = u - y & x > 0, y > 0 \\ u(y^2, y) = y & y > 0 \end{cases}$$

May a solution exist in a neighborhood of the origin?

References

[1] Kenneth S. Miller, *Partial Differential Equations in Engineering Problems*, Dover Publications inc. 2020.

[2] Walter A. Strauss, *Partial Differential Equations An Introduction*, John Wiley and Sons Ltd, 2008

[3] Sandro Salsa, *Partial Differential Equations in Action, From Modelling to Theory*, Springer, 2016.

[4] Lawrence C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, 2010.