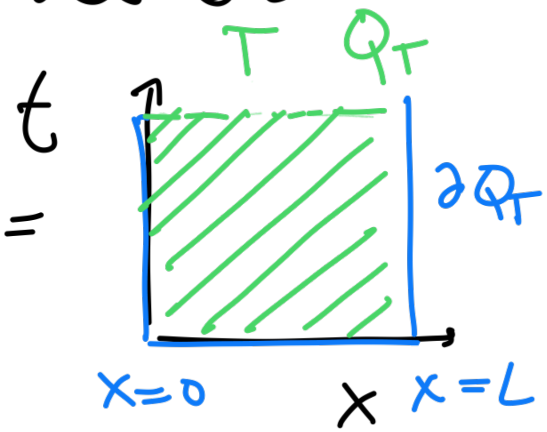
General idea: Heat flows from the higher temperature to the lower temperature.

$\Rightarrow$ a solution of the homogeneous heat equation should achieve its maximum/minimum values on

$$\partial Q_T = \overline{Q}_T - Q_T$$

$Q_T =$



$$\overline{Q}_T = \partial Q_T \cup Q_T$$

$$C^{2,1}(Q_T) = \{ u : Q_T \to \mathbb{R},\ u,\ D_x u,\ D_x^2 u,\ u_t \in C(Q_T) \}$$

Theorem (Maximum principle)

$u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ solves the heat equation on Q_T

then

(i) $\max\limits_{\overline{Q}_T} u = \max\limits_{\partial Q_T} u$

$\min\limits_{\overline{Q}_T} u = \min\limits_{\partial Q_T} u$

Weak Max principle

(ii) if U is connected and $\exists\ (x_0, t_0) \in Q_T$ such that

$$u(x_0, t_0) = \max_{\overline{Q}_T} u(x, t)$$

then temperature $u(x,t)$ is constant in $\overline{Q}_{t_0}$

Strong max principle

Proof (weak principle)

let us assume $u(x,t)$ is a solution

define $\overbrace{v(x,t) = u(x,t) - \varepsilon t}$

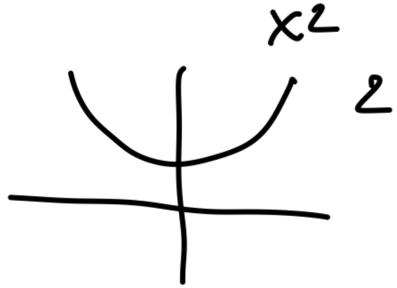
$$\underbrace{\frac{\partial v}{\partial t} - D\Delta v}_{} = \underbrace{\frac{\partial u}{\partial t} - D\Delta u}_{} - \varepsilon = \underbrace{-\varepsilon}_{} < 0 \qquad (*)$$

Step 1 let us first prove $\max_{\overline{Q_T}} v(x,t) = \max_{\partial Q_T} v(x,t)$

Proof let us assume $\exists\ (x_0,t_0) \in Q_T$

s.t (x_0,t_0) is a max of $v(x,t)$ over $\overline{Q_T}$

1) $v(x,t): Q_T \to \mathbb{R}$



if (x_0, t_0) is a local optimum of v

$\nabla v(x_0, t_0) = 0$

2) if $v(x,t)$ twice continuously differentiable

(x_0, t_0) stationary point

and local maximum $\to \nabla^2 v(x_0, t_0) \preceq 0$

(diagonal entries of the Hessian are ≤ 0)

together 1) + 2) $\Rightarrow$ $v_{xx}(x_0, t_0) \leq 0$

$$v_x(x_0, t_0) = 0$$

$$\lim_{h \to 0} \frac{v(x_0, t_0) - v(x_0, t_0 - h)}{h} \geq 0 \to v_t(x_0, t_0) \geq 0$$

$$\left.\begin{array}{l} v_{xx}(x_0,t_0) \le 0 \\ v_t(x_0,t_0) \ge 0 \end{array}\right\} \to v_t - D\Delta v \ge 0 \to \text{contradiction with } (*)$$

$$\to \quad \max_{\overline{Q_T}} v(x,t) = \max_{\partial Q_T} v \le \max_{\partial Q_T} u(x,t)$$

$$v(x,t) = u(x,t) - \varepsilon t$$

$$\lim_{\varepsilon \to 0} \quad \max_{\overline{Q_T}} u(x,t) - \varepsilon t \le \lim_{\varepsilon \to 0} \max_{\partial Q_T} u(x,t)$$

$$\max_{\overline{Q_T}} u(x,t) \le \max_{\partial Q_T} u(x,t)$$