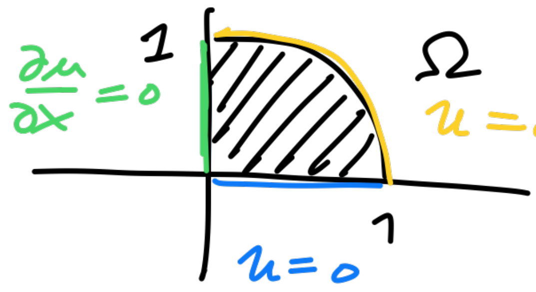
Recitation 5



Question 1

$$u_{xx} + u_{yy} = f(x,y)$$

in $\Omega = \{ (x,y), \ x>0, \ y>0 \quad x^2+y^2<1 \}$

$$\begin{cases} u = 0 & y = 0 & 0 < x < 1 \\ \dfrac{\partial u}{\partial x} = 0 & x = 0 & 0 < y < 1 \\ u = 0 & x > 0 \quad y > 0 & x^2+y^2 = 1 \end{cases}$$

Solution: → cylindrical coordinate

→ separation of variable

→ Fourier Series on $f(r,\theta)$

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = f(r,\theta)$$

$$u(r,0) = 0 \qquad 0 \le r < 1$$

$$u_\theta(r, \pi/2) = 0 \qquad 0 \le r < 1$$

$$u(1,\theta) = 0 \qquad 0 \le \theta < \pi/2$$

$$u(r,\theta) = X(r)\, Y(\theta)$$

$$\rightarrow \quad X''(r)\, Y(\theta) + \frac{1}{r} X'(r)\, Y(\theta) + \frac{1}{r^2} X(r)\, Y''(\theta) = f(r,\theta)$$

$$\frac{X''(r)r^2 + r\,X'(r)}{X(r)} = \frac{-Y''(\theta)}{Y(\theta)} = \lambda$$

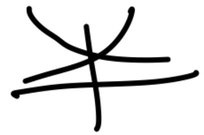
$$X''(r)r^2 + rX'(r) - \lambda X(r)$$

$$r = e^s \rightarrow \qquad X(r) = V(\log r) \qquad X(r) = V(s = \log(r))$$

$$\rightarrow \quad \frac{d}{dr}X(r) = \frac{dV}{ds}(s)\cdot\frac{1}{r}$$

$$\frac{d^2}{dr^2}X(r) = \frac{d^2V}{ds^2}(s)\frac{1}{r^2} + \frac{dV}{ds}\cdot\frac{-1}{r^2}$$

$$\frac{V''(s) \quad - V'(s) + V'(s)}{V(s)} = \overbrace{\frac{V''(s)}{V(s)}} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda$$



i) $Y''(\theta) = -\lambda \quad (\theta)$

$Y(\theta) = A\theta + B$ $\qquad V(0) = 0 \qquad V_\theta(\pi/2) = 0$

ii) $Y(\theta) = Ae^{\mu\theta} + Be^{-\mu\theta}$ $\qquad V(0) = 0 \Rightarrow A = -B$

$V_\theta(\pi/2) = 0 \Rightarrow A\mu(e^{\mu\theta} + e^{-\mu\theta})$

$A = 0$

(iii) $Y(\theta) = Ae^{i\mu\theta} + Be^{-i\mu\theta}$

$V(0) = 0 \Rightarrow A = -B$

$V_\theta(\pi/2) = 0 \Rightarrow (2\sin\mu\theta)_\theta = 2\mu\cos\mu\theta$

$2\mu\cos\mu\pi/2 = 0 \Rightarrow \mu\pi/2 = \frac{\pi}{2} + k\pi$

$\mu = 1 + 2k$

From this we get $\lambda_k = (1+2k)^2$

$$U(s,\theta) = \sum_{k=0}^{\infty} A_k \sin(1+2k)\theta \cdot v(s)$$

$$\Delta U = f(r,\theta) = f(e^s,\theta)$$

$$\sum_{k=0}^{\infty} v''(s) A_k \sin((1+2k)\theta) + \sum_{k=0}^{\infty} v(s)(1+2k)^2 A_k \sin((1+2k)\theta)$$

$$= f(e^s,\theta)$$

$$f(s,\theta) = \sum_{l=1}^{\infty} F_l \sin(l\theta)$$

$$F_\ell = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s,\theta) \sin \ell\theta \, d\theta$$

We then match F_ℓ with

$$\boxed{V''(s) A_k + V(s)(1+2k)^2 A_k = F_{2k+1}}$$

$r^{-(1+2k)}$

$\uparrow$

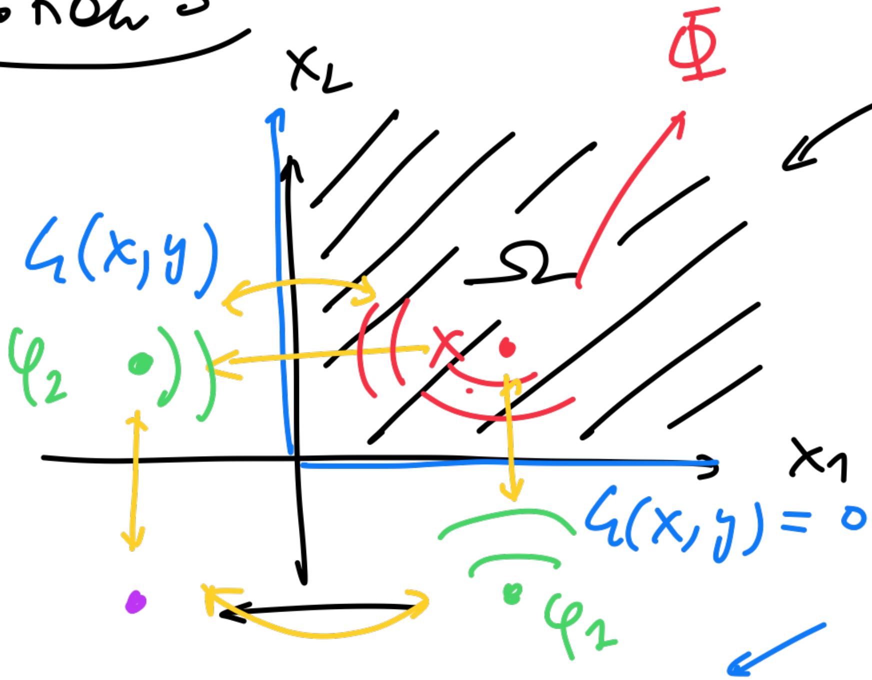
$s = \log r$

→ Homogeneous solution → $V_H(s) = \underline{e^{(1+2k)s}} + \cancel{e^{-(1+2k)s}}$

Complete solution : $V(s) = V_H(s) + V_{particular}(s)$

→ then apply remaining BC.

Question 3



Green function for $\{(x_1, x_2) \;\; x_1 > 0, \; x_2 > 0\}$

$\partial\Omega = \{x_1 = 0, \; x_2 = 0\}$

$$G(x,y) = \Phi(x-y) - \varphi(y)$$

$\Phi(y-x)$

$$-\frac{1}{2\pi}\log(|y-x|) + \frac{1}{2\pi}\log\left(\left|y - \begin{pmatrix} 0 \\ -x_2 \end{pmatrix}\right|\right) + \frac{1}{2\pi}\log\left(\left|y - \begin{pmatrix} -x_1 \\ 0 \end{pmatrix}\right|\right)$$

$$-\frac{1}{2\pi}\log\left(\left|y - \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}\right|\right)$$

Recitation 5

Question 6

Find the one dimensional Green's function for the interval $(0, l)$. the three properties defining it can be restated as

(i) It solves $G''(x; x_0) = 0$ for $x \neq x_0$

(ii) $G(x; x_0) = 0$ if $x = 0$ or $x = l$

(iii) $G(x; x_0)$ is continuous at x_0 and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0

$$-\Delta G(x, x_0) = \delta(x - x_0) \text{ in } D \quad (*)$$

$$G = 0 \text{ on } \partial D$$

$$G''(x, x_0) = 0 \quad \text{on } [0, x_0)$$

$$G''(x, x_0) = 0 \quad \text{on } (x_0, \ell]$$

$$G(x, x_0) = C_1 + C_2 x \quad \text{if } x \in [0, x_0)$$

$$= C_1' + C_2' x \quad \text{if } x \in (x_0, \ell]$$

$$G(x, x_0) = 0 \quad \text{for } x = 0 \rightarrow C_1 = 0$$

$$G(x, x_0) = 0 \quad \text{for } x = \ell \rightarrow C_1' + C_2' \ell = 0 \rightarrow \overline{C_1' = -C_2' \ell}$$

Using continuity we get

$$C_2 x_0 = -C_2' \ell + C_2' x_0$$

$$C_2' = C_2 \frac{x_0}{(x_0-\ell)} \rightarrow \quad C_1' = -\frac{C_2 x_0}{(x_0-\ell)} \ell \qquad C_1 = 0$$

We are left with C_2 which can be determined by solving Laplace's equation (*)

$$-\int_{x_0-\varepsilon}^{x_0+\varepsilon} G''(x-x_0)\, dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x-x_0)\, dx$$

$$-\underbrace{\left|\frac{dG_1}{dx}\right|}_{x_0+\varepsilon} + \left|\frac{dG_1}{dx}\right|_{x_0-\varepsilon} = 1$$

$$-C_2' + C_2 = 1$$

$$-C_2\,\frac{x_0}{x_0-l} + C_2 = 1 \;\rightarrow\; C_2\left(1-\frac{x_0}{x_0-l}\right) = 1$$

$$C_2 = \frac{1}{\left(1-\frac{x_0}{x_0-l}\right)}$$

$$c_1 = 0 \qquad c_2 = \frac{1}{\left(1 - \frac{x_0}{x_0 - l}\right)} \qquad c_2' = \frac{x_0}{x_0 - l} \cdot \left(\frac{1}{1 - \frac{x_0}{x_0 - l}} \right)$$

$$c_1' = - c_2' l$$

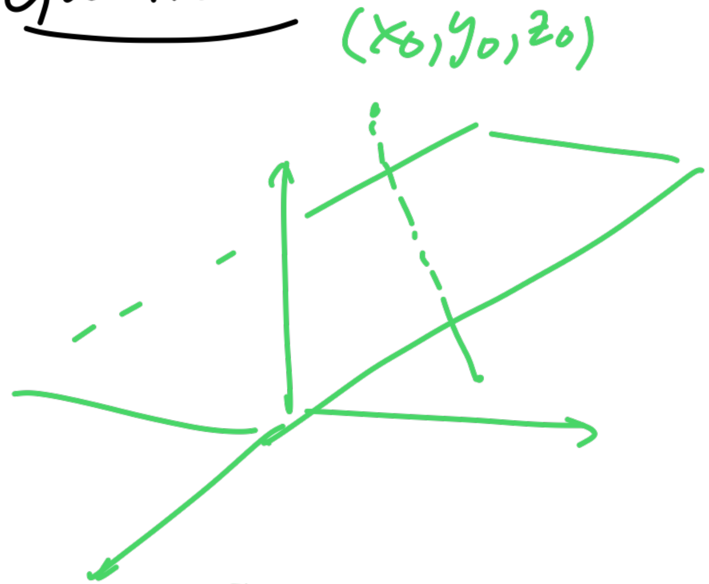
$$G(x, x_0) = \begin{cases} x \, \frac{(x_0 - l)}{-l} = \frac{l - x_0}{l} \, x & \text{if } x \in [0, x_0) \\ \frac{x_0}{x_0 - l} \left(\frac{1}{1 - \frac{x_0}{x_0 - l}} \right) x - \frac{x_0}{x_0 - l} \left(\frac{1}{1 - \frac{x_0}{x_0 - l}} \right) & \text{if } x \in (x_0, l] \end{cases}$$

$$\frac{x_0}{x_0 - l} \, \frac{x_0 - l}{-l} \, x - \frac{x_0}{x_0 - l} \, \frac{x_0 - l}{-l}$$

$$G(x,x_0) = \begin{cases} \dfrac{l-x_0}{l}\,x & x \in [0,x_0) \\[2ex] \dfrac{-x_0}{l}\,x + x_0 & x \in (x_0,l] \end{cases}$$

$$\frac{l-x_0}{l}\,x_0 \overset{!}{=} -\frac{x_0^2}{l} + x_0$$

Question 7

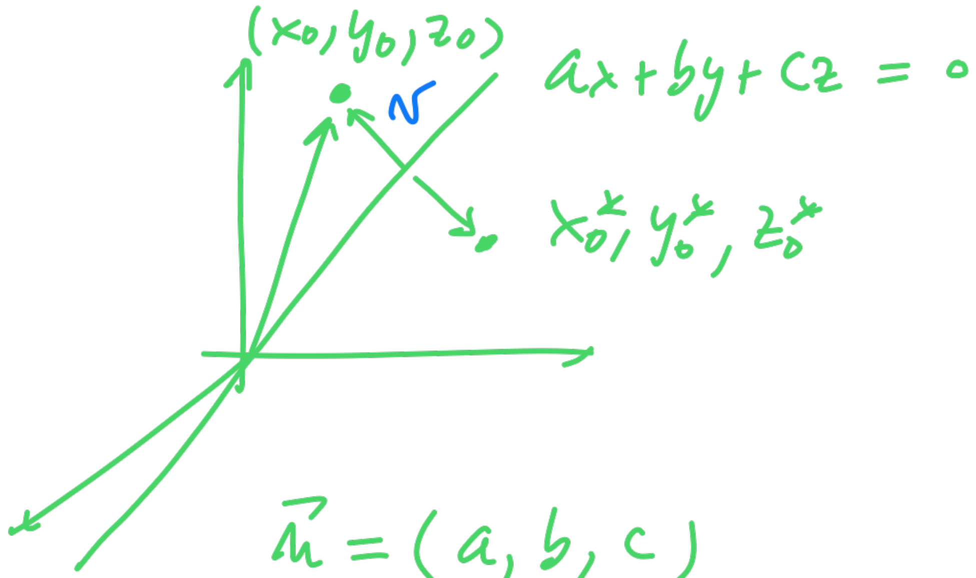


$$G(x, x_0) = \Phi(x, y, z; x_0, y_0, z_0)$$

$$\Phi(x, y, z; x_0, y_0, z_0) = \frac{-1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$

Question: Where can we put x_0^* such that

$$G(x, x_0) = 0 \text{ on } \pi = \{(x, y, z) \mid ax+by+cz=0\}$$



$\vec{n} = (a, b, c)$

projection of (x_0, y_0, z_0) onto (a, b, c)

is given $(x_0 a + y_0 b + z_0 c)$

From this, we get v as

$$v = (x_0 a + y_0 b + z_0 c) \cdot \frac{(a, b, c)}{\sqrt{a^2+b^2+c^2}}$$

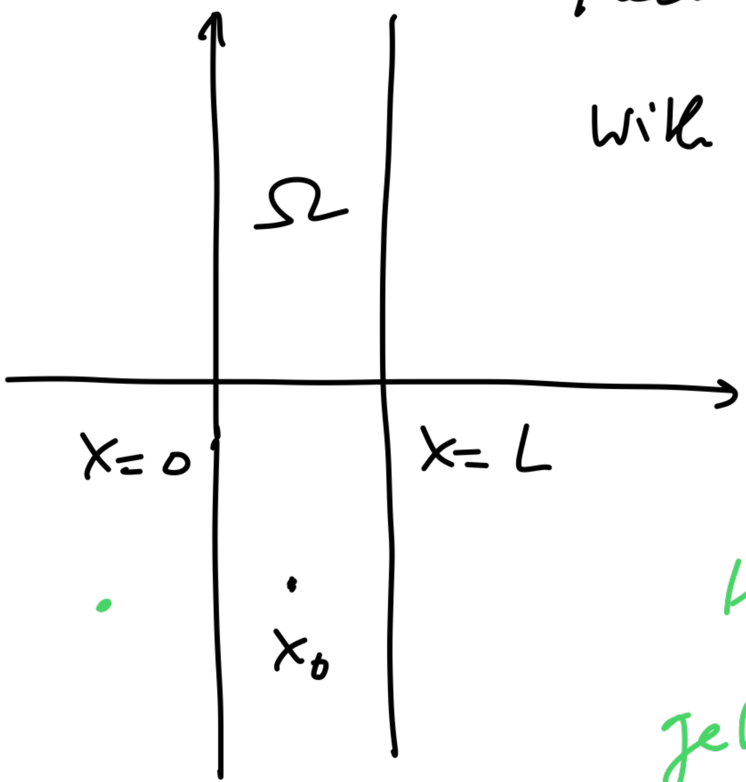
To get the position of the negative source, we subtract twice the vector v from (x_0, y_0, z_0)

$$(x_0^*, y_0^*, z_0^*) = (x_0, y_0, z_0) - 2\frac{(a, b, c)}{\sqrt{a^2+b^2+c^2}} \cdot (ax_0 + by_0 + cz_0)$$

From this we can define our Green function as

$$G(x, x_0) = \Phi(x, y, z; x_0, y_0, z_0) - \Phi(x, y, z, x_0^*, y_0^*, z_0^*)$$

Question 4.2



Green function in Ω

with $G(x, x_0) = 0$ at $x=0$ and $x=L$

let us start with $\Phi(x-x_0)$

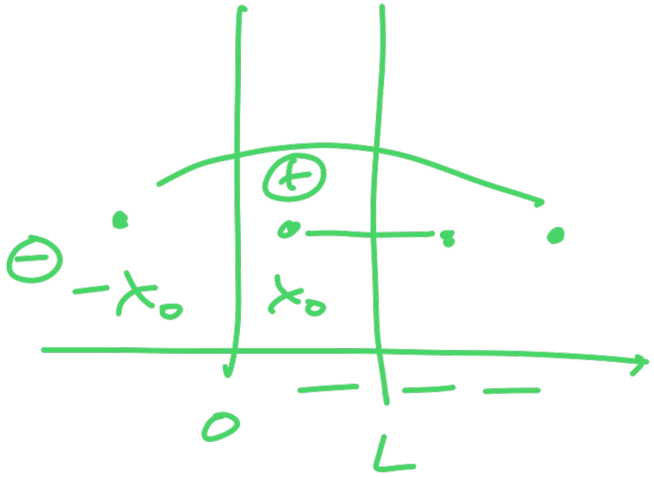
then let us add a source to

get cancelation on $x=0$

first source should be added at $-x_0 = x_0^*$

Issue: we have cancelation on $x=0$ but not on $x=L$

at this stage we have



$$\Phi((x-x_0)) - \Phi(x+x_0)$$

$\rightarrow$ to get cancelation at $x=L$

we add 2 more sources at

$$\begin{cases} 2L-x_0 \\ L+x_0 \end{cases}$$

we now have $\Phi\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) - \Phi\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -x_0 \\ y_0 \end{pmatrix}\right)$

$$- \Phi\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2L-x_0 \\ y_0 \end{pmatrix}\right) + \Phi\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} L+x_0 \\ y_0 \end{pmatrix}\right)$$

$$G(x, x_0) = \sum_{k=-\infty}^{\infty} \Phi\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_n \\ y_0 \end{pmatrix}\right) - \Phi\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_n^* \\ y_0 \end{pmatrix}\right)$$

$$x_n^* = nL - x_0 \qquad -$$

$$x_n = nL + x_0 \qquad +$$