$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = \varphi(x) \end{cases}$$

$$u(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4Dt}\right) \varphi(y)\, dy \leftarrow$$

$$= \int_{-\infty}^{\infty} \Phi(x-y)\, \varphi(y)\, dy$$

$$\to \int_{-\infty}^{\infty} \partial_t \Phi(x-y)\, \varphi(y) + \Phi_{xx}(x-y)\, \varphi(y)\, dy$$

$$\int \left[\partial_t \Phi(x-y) + \Phi_{xx}(x-y)\right] \varphi(y)\, dy = 0$$

$$\lim_{t \to 0} \int_{-\infty}^{\infty} \Phi(x-y)\, \varphi(y) = \varphi(x) \qquad \lim_{t \to 0} \Phi(x-y,t) = \delta(x-y)$$

$$\int_{-\infty}^{\infty} \underline{\delta(x-y)\,\varphi(y)}\, dy = \varphi(x)$$

$$\varphi(x) = \varphi(-x) \Rightarrow u(x,t) = u(-x,t)$$

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{(4\pi D)^{1/2}} \exp\left(-\frac{(x-y)^2}{4Dt}\right) \varphi(y)\, dy$$

$$u(-x,t) = \int_{-\infty}^{\infty} \frac{1}{(4\pi D)^{1/2}} \exp\left(-\frac{(-x-y)^2}{4Dt}\right) \varphi(y)\, dy$$

$$y \leftarrow -y$$

$$= \int_{-\infty}^{\infty} \frac{1}{(4\pi D)^{1/2}} \exp\left(-\frac{(y-x)^2}{4Dt}\right) \varphi(-y)\, dy$$

$$\varphi(-y) = \varphi(y)$$

Question 15

(*) $u_t - k u_{xx} + bu = 0 \qquad u(x,0) = g(x)$

$$u(x,t) = e^{-bt} v(x,t)$$

Substitute $u(x,t)$ in (*), gives

$$-b \cancel{e^{-bt}} v + \cancel{e^{-bt}} v_t - k \cancel{e^{-bt}} v_{xx} + b \cancel{e^{-bt}} v = 0$$

$$v_t - k v_{xx} = 0$$

$$v(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4Dt}\right) \cdot g(y)\, dy$$

$$u(x,t) = \frac{e^{-bt}}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4Dt}\right) g(y)\, dy$$

Question 9

$$v(x,t) = t^{\alpha} w\left(\frac{x}{t^{\beta}}\right)$$

$$v_t = v_{xx} + (v^2)_x \quad (*)$$

$$v_t = \alpha t^{\alpha-1} w\left(\frac{x}{t^{\beta}}\right) + t^{\alpha} w'\left(\frac{x}{t^{\beta}}\right) \cdot \frac{x}{t^{\beta+1}}(-\beta)$$

$$v_x = t^{\alpha} w'\left(\frac{x}{t^{\beta}}\right) \cdot \frac{1}{t^{\beta}}$$

$$v_{xx} = t^{\alpha} w''\left(\frac{x}{t^{\beta}}\right)\left(\frac{1}{t^{\beta}}\right)^2$$

Substituting thse into $(*)$ we get

$$v_t = v_{xx} + (v^2)_x$$

$$\alpha t^{\alpha-1} w\left(\underbrace{\frac{x}{t^\beta}}_{\xi}\right) - \beta t^\alpha \frac{x}{t^{\beta+1}} w'\left(\underbrace{\frac{x}{t^\beta}}_{\xi}\right)$$

$$= t^\alpha w''\left(\frac{x}{t^\beta}\right)\left(\frac{1}{t^\beta}\right)^2 + 2 t^\alpha w\left(\frac{x}{t^\beta}\right) \cdot t^\alpha w'\left(\frac{x}{t^\beta}\right) \cdot \frac{1}{t^\beta}$$

$$\frac{x}{t^\beta}$$

$$\alpha t^{\alpha-1} w(\xi) - \beta t^{\alpha-\beta-1} w'(\xi)\, x$$

$$= t^{\alpha-2\beta} w''(\xi) + 2 t^{2\alpha-\beta} w'(\xi) w(\xi)$$

$$t^{\alpha-\beta-1} = t^{-\beta}$$

$$t^{\alpha-2\beta} = 1 = t^{2\alpha-\beta}$$

$$t^{\alpha-1} = 1$$

$\alpha = 1$

$1 - 2\beta$

$2 - \beta$

simplifying by $t^{\alpha-1}$

we get

$$\alpha W(\xi) - \beta \frac{x}{t^{\beta}} W'(\xi) = t^{-2\beta+1} W''(\xi) + 2\, t^{\alpha-\beta+1} W'(\xi) W(\xi)$$

$\beta = 1/2$

$\alpha = -1/2$

$$\alpha W(\xi) - \beta \xi W'(\xi) = W''(\xi) + 2W'(\xi) W(\xi)$$

$$-\frac{1}{2} W(\xi) - \frac{1}{2} \xi W'(\xi)$$

$$-\frac{1}{2} \left(W(\xi) \cdot \xi \right)' = \quad (W^2)' + W''(\xi)$$

integrate w.r.t ξ

$$-\frac{1}{2}\left(w(\xi)\,\xi\right) = w^2 + w' + \underbrace{c}_{}\overset{}{=}0$$

Which is a first order non linear ODE with variable

ODE

$$w(\xi) = \frac{1}{Z(\xi)}$$

$$-\frac{1}{2}\,\frac{1}{Z}\,\xi = \frac{1}{Z^2} + \frac{-1}{Z^2}\cdot Z'$$

$$-\frac{1}{2} Z\xi = 1 - Z'$$

$$Z' = 1 + \frac{Z(\xi)}{2}\xi$$

Theorem (Variable coefficients)

if functions a, b continuous then

$$y' = a(t)y + b(t)$$

has infinitely many solutions of the form

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)} b(t)\, dt$$

where $A(t) = \int a(t)\, dt$ and $c \in \mathbb{R}$