Theorem Let $f \in C^2(\mathbb{R}^n)$ with compact support. Let u be defined as the Newtonian potential

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

then u is the unique solution in $\mathbb{R}^n$ of $\Delta u = -f$ belonging to $C^2(\mathbb{R}^n)$ and vanishing at ∞.

Proof u, v 2 solutions vanishing at ∞

then from an application of Harnack's inequality to $u - v$, we get

$$\max(u-v) \leq C \cdot \min(u,v)$$

$$\left.\begin{array}{l}\text{as well as for } v-u \\ + \; u,v \text{ vanishing at } \infty\end{array}\right\} \Rightarrow u = v$$

$\rightarrow$ From this we have uniqueness, we are left with exhibiting a solution.

Let us take u defined as

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \quad \text{and let us show} \quad \Delta u = -f$$

$$\boxed{\Phi_n(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & n=2 \\ \frac{1}{n(n-2)V_n}\frac{1}{|x|^{n-2}} & n\geq 3 \end{cases}}$$

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y)\, f(x-y)\, dy$$

let us start by computing the partial derivatives of u

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x+he_i-y) - f(x-y)}{h}\right] dy$$

To move the limit inside the integral, we need to satisfy 2 conditions:

(i) uniform convergence of the sequence

$$g_h = \frac{u(x+he_i) - u(x)}{h}$$

(ii) finite integration domain (given from the assumption supp(f) compact)

For (i) for 1st order derivatives we can prove uniform convergence by means of a Taylor expansion

To see this, define

$$\left| \frac{\partial f}{\partial x_i} - \frac{f(x+he_i-y) - f(x-y)}{h} \right|$$

using Taylor approximation

$$\left| \frac{\partial f}{\partial x_i} - \frac{f(x+he_i-y) - f(x-y)}{h} \right| \leq h \sup_{\mathrm{supp}(f)} |D^2 f| \quad (*)$$

For second order derivatives, we cannot use Taylor

From the mean value theorem, $\exists\, t \in [0,1]$ s.t

$$\frac{1}{h}\left(\frac{\partial f}{\partial x_i}(x+he_j-y) - \frac{\partial f}{\partial x_i}(x-y) \right) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x+the_j-y)$$

From this

$$\left| \frac{1}{h}\left[\frac{\partial f}{\partial x_i}(x+he_j-y) - \frac{\partial f}{\partial x_i}(x-y) \right] - \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y) \right|$$

$$= \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x+the_j-y) - \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y) \right|$$

Mean value theorem holds for every x

Continuity of $\frac{\partial^2 f}{\partial x_i \partial x_j}$ + compact support of f

$\Rightarrow$ uniform continuity of $\frac{\partial^2 f}{\partial x_i \partial x_j}$ $(*)$

$(*) \Rightarrow \quad \forall \varepsilon \ \exists H \ \text{s.t} \ \forall h < H \quad \forall x$

$$\left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x + the_j - y) - \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y) \right| < \varepsilon$$

and hence

$$\left| \left[\frac{\partial f}{\partial x_i}(x + he_j - y) - \frac{\partial f}{\partial x_i}(x-y) \right] \frac{1}{h} - \frac{\partial^2 f}{\partial x_i \partial x_j} \right| < \varepsilon \quad (**)$$

$\forall x, \ h < H$

$\Rightarrow$ together $(*)$ and $(**)$

$$\lim_{h \to 0} \left[\frac{\partial u}{\partial x_i}(x + he_j - y) - \frac{\partial u}{\partial x_i}(x-y) \right]$$
$$= \int_{\mathbb{R}^n} \Phi(y) \lim_{h \to 0} \left[\frac{\partial f}{\partial x_i}(x + he_j - y) - \frac{\partial f}{\partial x_i}(x-y) \right] dy$$

$$\Rightarrow \underline{\frac{\partial^2 u}{\partial x_i \partial x_j}} = \int_{\mathbb{R}^n} \Phi(y) \underline{\frac{\partial^2 f}{\partial x_i \partial x_j}} \, dy$$

To show $u \in C^2$ we are left with showing continuity of

$$\int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i \partial x_j} dy$$

We will view the integral as an improper integral letting

$$\int_{\mathbb{R}^n} \Phi(y)\, \partial_{x_i x_j} f(x-y)\, dy = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \partial_{x_i} \partial_{x_j} f(x-y)\, dy$$

using $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is uniformly continuous (continuity + compact set)

$$\Rightarrow \forall \varepsilon' \ \exists \delta \ \text{s.t} \ \underbrace{|z| < \delta}$$

$$\Big| \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y)\, \partial_{x_i x_j} f(x-y+z)\, dy$$

$$- \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y)\, \partial_{x_i x_j} f(x-y)\, dy \Big|$$

$$< \Big| \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y) \left[\partial_{x_i x_j} f(x-y+z) - \partial_{x_i x_j} f(x-y) \right] dy$$

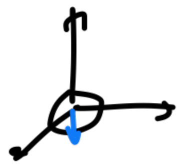
$$< \Big| \lim_{\varepsilon \to 0} \sup_{\mathbb{R}^n \setminus B_\varepsilon(0)} \underbrace{\left| \partial_{x_i x_j} f(x-y+z) - \partial_{x_i x_j} f(x-y) \right|} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y)\, dy \Big|$$

$$< \varepsilon' \Big| \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y)\, dy \Big|$$

From this we have

$$\left| \frac{\partial^2 u}{\partial x_i \partial x_j}(x+z) - \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \right| < \varepsilon' \rightarrow u \in C^2$$

To conclude, we need to show that our solution u satisfies $\Delta u = -f$



$$\Delta u(x) = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy$$

$$= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \Delta_y f(x-y)\, dy$$

$$= \int_{\partial B(0,\varepsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \nu}(x-y)\, dS - \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} D\Phi \cdot Df(x-y)\, dy$$

$$(*) \leq \sup_{\partial B(0,\varepsilon)} |Df| \left| \int_{\partial B(0,\varepsilon)} \Phi(y)\, dS \right| \leq \begin{cases} C \log \varepsilon \cdot \varepsilon \\ C \dfrac{1}{\varepsilon^{n-2}} \varepsilon^{n-1} \end{cases}$$

using $\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \log(|x|) & n=2 \\ \dfrac{1}{n(n-2)V_n} \dfrac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$

taking $\lim_{\varepsilon \to 0} (*) = 0$

$$(*) = - \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} D\Phi \cdot Df(x-y)\, dy$$

Applying integration by parts a second time,

we get

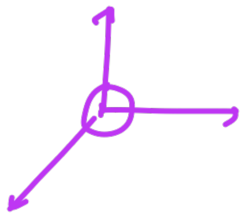
$$-\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \overbrace{D\Phi}^{f}\cdot \overbrace{Df}^{g'}(x-y)\,dy = \underbrace{-\int_{\partial B_\varepsilon(0)} \frac{\partial\Phi}{\partial\nu}\, f(x-y)\,dS}_{(*)} + \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \underbrace{\Delta\Phi}_{=0}\cdot f(x-y)\,dy$$

$$-\int_{\partial B_\varepsilon(0)} \frac{\partial\Phi}{\partial\nu}\, f(x-y)\,dS =$$

we $D\Phi(y) = \begin{cases} -\dfrac{1}{2\pi}\dfrac{y}{|y|} = -\dfrac{y}{2\pi|y|^2} \\[2ex] \dfrac{1}{n(n-2)V_n}(2-n)\dfrac{y}{|y|^n} \end{cases}$

$$|y| = \overbrace{\sqrt{y_1^2+\dots y_n^2}}^{\sqrt{|y|^2}}$$

$$= -\frac{1}{nV_n}\underbrace{\frac{y}{|y|^n}}$$



$$\vec{\nu} = -\frac{y}{|y|}$$

$$\frac{\partial\Phi}{\partial\nu} = D\Phi\cdot\vec{\nu} = \frac{|y|^2}{nV_n|y|^{n+1}}$$

$$= \frac{1}{nV_n\underbrace{|y|}^{n-1}}$$

Substituting this in (*)

$$-\int_{\partial B(0,\varepsilon)} \underbrace{\frac{\partial \Phi}{\partial \nu}} \cdot f(x-y)\, dS = -\overbrace{\frac{1}{n V_n \varepsilon^{n-1}} \underbrace{\int_{\partial B(0,\varepsilon)}}} f(x-y)\, dy$$

$$\lim_{\varepsilon \to 0} -\frac{1}{n V_n \varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y)\, dy$$

$$= \underbrace{-f(x)}$$

Before introducing the notion of Green function let us introduce Green's formulas

$$\boxed{\begin{array}{l}\text{Theorem (Gauss-Green)} \\ \quad \text{Suppose } u \in C^1(\bar{U}) \\ \text{(i)} \displaystyle\int_U u_{x_i}\, dx = \int_{\partial U} u \cdot \nu^i\, dS \\ \text{(ii)} \displaystyle\int_U \nabla \cdot u\, dx = \int_{\partial U} u \cdot \nu\, dS \end{array}}$$

<u>Integration by parts</u> follows from an application of (i) to the product (uv)

$$\boxed{\int_U (uv)_{x_i}\, dx = \int_U u_{x_i} v + \int_U u v_{x_i} = \int_{\partial U} (uv) \cdot \nu^i\, dS}$$

From (*) taking $u = u_{x_i}$ $\quad v = 1$

GF1

$$\boxed{\int_U \Delta u \, dx = \int_{\partial U} u_{x_i} \cdot \nu^i \, dS = \int_{\partial U} \frac{\partial u}{\partial \nu} \, dS}$$

From (*) taking $v = v_{x_i}$

GF2

$$\boxed{\int_U Du \, Dv \, dx + \int_U u \, \Delta v \, dx = \int_{\partial U} \frac{\partial v}{\partial \nu} u \, dS}$$

For GF3 we interchange u and v in GF2

and subtract

$$\int_U Du \, Dv \, dx + \int_U v \Delta u = \int_{\partial U} v \frac{\partial u}{\partial \nu} \, dS$$

GF3

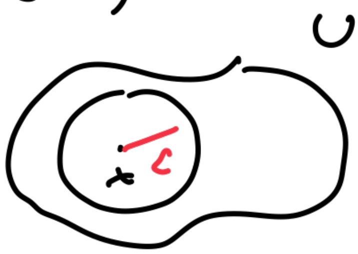
$$\Rightarrow \boxed{\int_U (u \Delta v - v \Delta u) \, dx = \int_{\partial U} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS}$$

From this take $u \in \underline{C^2(\bar{U})}$ $\quad x \in U$ $\quad \varepsilon > 0$

such that $B_\varepsilon(x) \subset U$

let us apply GF3 to $V_\varepsilon = U \setminus B_\varepsilon(x)$

with $v(y) = \underline{\Phi(x-y)}$

$V_\varepsilon =$ 

$$\int_{V_\varepsilon} u \, \underbrace{\cancel{\Delta \Phi(y-x)}}_{=0} \, dy - \int_{V_\varepsilon} \Delta u \, \Phi(x-y) \, dy$$

$$= \int_{\partial V_\varepsilon} u(y) \frac{\partial \Phi}{\partial \nu} \, dS - \int_{\partial V_\varepsilon} \Phi(y-x) \frac{\partial u}{\partial \nu} \, dS$$

$$\partial V_\varepsilon = \partial U \cup \partial B(x,\varepsilon)$$

$$-\int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu}(y) \, dS \leq C \varepsilon^{n-1} \sup |\Phi(y)| \leq \begin{cases} C \varepsilon \log \varepsilon & n=2 \\ C \varepsilon & n \geq 3 \end{cases}$$

$$\int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu} \, dS = ⨍_{\partial B(x,\varepsilon)} u(y) \, dS \xrightarrow[\text{as } \varepsilon \to 0]{} u(x)$$

From this we recover the integral formulation

$$u(x) = \int_U \Delta u \, \Phi(x-y) \, dy + \int_{\partial U} u \frac{\partial \Phi}{\partial \nu} \, dS - \int_{\partial U} \Phi \frac{\partial u}{\partial \nu} \, dS \quad (**)$$

$$\begin{cases} \Delta u = f & \text{on } U \\ u = g & \text{on } \partial \Omega \end{cases}$$

if we could find a function $\varphi(y)$ s.t

$$\begin{cases} \Delta \varphi = 0 & \text{on } U \\ \varphi(y) = \Phi(x-y) & \text{on } \partial U \end{cases}$$

then if apply GF III to u and φ,

$$\int_U \Delta u \, v - \Delta v \, u = \int_{\partial U} v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \, dS$$

take $v = \varphi$

$$\int_U \Delta u \, \varphi - \overbrace{\Delta \varphi}^{=0} \, u = \int_{\partial U} \underbrace{\varphi}_{=\Phi(x-y)} \frac{\partial u}{\partial \nu} - u \frac{\partial \varphi}{\partial \nu} \, dS \quad (*)$$

$$\int_{\partial U} \overbrace{\varphi}^{=\Phi(x-y)} \frac{\partial u}{\partial \nu} = \int u \frac{\partial \varphi}{\partial \nu} \, dS + \int_U \Delta u \, \varphi$$

Substituting this in $(**)$

$$u(x) = \int_U \Delta u \, (\Phi - \varphi) \, dx - \int_{\partial U} u \frac{\partial (\Phi - \varphi)}{\partial \nu} \, dS$$

Now we can let $G(x, y) = \Phi(x-y) - \varphi(y)$

and we call this function the Green function

$$u(x) = \int_U \Delta u \, G(x-y) \, dy - \int_{\partial U} u \frac{\partial G}{\partial \nu} \, dS$$

How can we find φ ? and hence $G(x,y)$

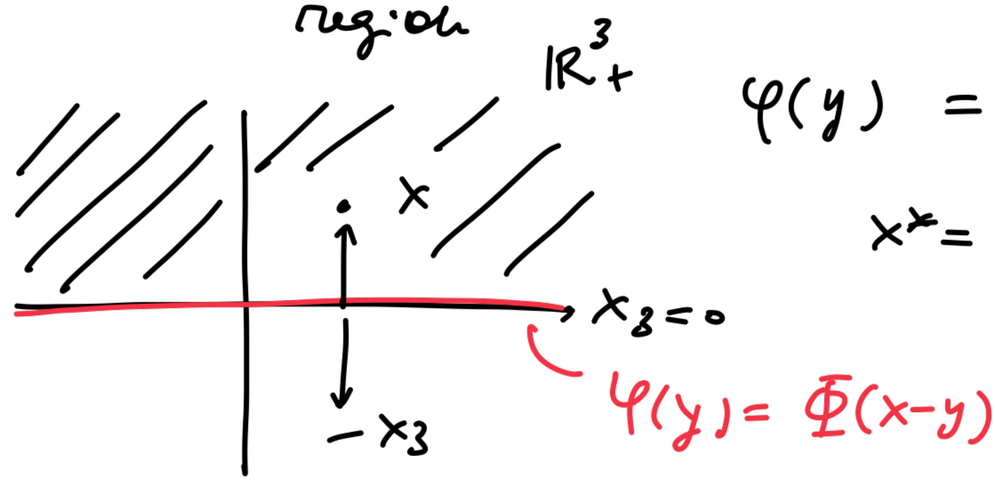
<u>Example 1</u> half space $\{x_3 > 0\}$

$\{(x_1, x_2, x_3),\ x_3 > 0\} = U$

$\partial U = \{(x_1, x_2, x_3 = 0)\}$

take $\varphi(y) = \Phi(y-x) \Rightarrow$ problem : singularity at x

$\rightarrow$ idea : move the singularity outside the region



$\varphi(y) = \Phi(y - x^*)$

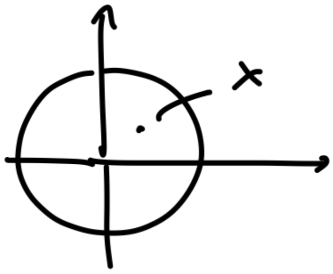
$x^* = (x_1, x_2, -x_3)$

From this the Green function for the $\{x_3 > 0\}$ half space can read as

$$G(x,y) = \Phi(x-y) - \Phi(y - x^*)$$

$x^* = (x_1, x_2, -x_3)$

Example 2 let us take $U = B(0,1)$ (radius 1 ball centred at 0, in $\mathbb{R}^3$



We look for a φ s.t

$$\Delta \varphi = 0 \text{ on } B(0,1)$$

$$\varphi = \Phi(x-y) \text{ on } \partial B(0,1)$$

$$\Phi(x-y) = \frac{1}{4\pi |x-y|}$$

find $\varphi(y) = \dfrac{q}{4\pi |y-x^*|}$ s.t

$$\frac{q}{4\pi |y-x^*|} = \frac{1}{4\pi |x-y|} \text{ on } |y| = 1$$

$$q^2 |x-y|^2 = |y-x^*|^2$$

$$= q^2\left(|x|^2 + |\overset{1}{\cancel{y}}|^2 - 2\langle x,y\rangle\right) = |\overset{1}{\cancel{y}}|^2 + |x^*|^2 - 2\langle x^*,y\rangle$$

$$q^2\left(|x|^2 - 2\langle x,y\rangle + 1\right) = |x^*|^2 - 2\langle x^*,y\rangle + 1$$

$$\underbrace{|x^*|^2 + 1 - q^2|x|^2 - q^2}_{\text{LHS only depends on } x} = 2\langle x^*,y\rangle - 2q^2\langle x,y\rangle$$

$$= \underbrace{2\langle y, (x^* - q^2 x)\rangle}_{\text{depends on } x,\, y}$$

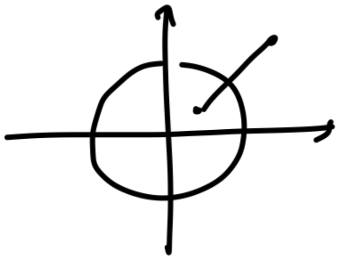
$$\Rightarrow \quad x^* = q^2 x$$

as well as $\quad q^4 |x|^2 + 1 - q|x|^2 - q^2 = 0$

$$|x|^2 (q^4 - q^2) + 1 - q^2 = 0$$

$$\underbrace{(q^2 - 1)}\underbrace{(|x|^2 q^2 - 1)} = 0$$

$$q = \frac{1}{|x|} \quad \Rightarrow \quad x^* = \frac{x}{|x|^2} \Big)$$ inversion through ∂B



$$G(x,y) = \underbrace{\Phi(x-y)}_{\frac{1}{4\pi|x-y|}} - \frac{1}{4\pi \left| x - \frac{x}{|x|^2} \right| |x|}$$