Recitation 8 Question 1

$$\Delta u + \frac{\omega^2}{c^2} u = 0 \qquad (*)$$

a) $u(x,\omega) = A(x,\omega) e^{i\omega\phi(x)}$

$$\Delta u \Rightarrow \frac{\partial}{\partial x_j} u(x,\omega) = \partial_j A \, e^{i\omega\phi(x)} + A i\omega \partial_j \phi \, e^{i\omega\phi}$$

$$\frac{\partial^2}{\partial x_j^2} = \partial_j^2 A \, e^{i\omega\phi} + \partial_j A \, i\omega \, \partial_j\phi \, e^{i\omega\phi}$$
$$+ \partial_j A \, i\omega \, \partial_j\phi \, e^{i\omega\phi} + A i\omega \, \partial_j^2\phi \, e^{i\omega\phi}$$
$$+ A (i\omega)^2 (\partial_j\phi)^2 e^{i\omega\phi}$$

Substituting this into $(*)$ and summing we get

$$\sum_j \Big(\partial_j^2 A e^{i\omega\phi} + \partial_j A\, i\omega\, \partial_j \phi\, e^{i\omega\phi} + \partial_j A\, i\omega\, \phi\, e^{i\omega\phi}$$

$$+ A i\omega\, \partial_j^2 \phi\, e^{i\omega\phi} + A \underbrace{(i\omega)^2} (\partial_j \phi)^2 e^{i\omega\phi} \Big)$$

$$+ \frac{\omega^2}{c^2} A e^{i\omega\phi}$$

Multiplying by $e^{-i\omega\phi}$, dividing by $A^2 \cdot \omega^2$, we get

$$\sum_j \Big(\cancel{\frac{\partial_j^2 A}{A\,\omega^2}} + \cancel{\frac{\partial_j A}{A} \frac{i}{\omega}} \partial_j \phi + \overbrace{\frac{\partial_j A\;\; i}{A\,\omega} \partial_j \phi} + \cancel{\frac{i}{\omega} \partial_j^2} \phi$$

$$\overbrace{- (\partial_j \phi)^2} \Big) = \frac{-1}{c^2}$$

$$\frac{i}{\omega} \Big(\sum_j \partial_j^2 \phi + \sum_j 2 \frac{\partial_j A}{A} \partial_j \phi \Big) + \Big(\frac{1}{c^2} - \sum_j (\partial_j \phi)^2 \Big) + \sum_j \frac{\partial_j^2 A}{A\,\omega^2} = 0$$

Taking the limit $\omega \to \infty$ we recover

$$\frac{1}{c^2} = \sum_j (\partial_j \phi)^2$$

Eikonal equation

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from here we use u to denote the phase

$$\begin{cases} \frac{1}{c^2} = |\nabla u|^2 = u_x^2 + u_y^2 \\ u(x,0) = -\sqrt{1+x^2} \end{cases}$$

$\Gamma(s) = (f(s), g(s))$ $\quad f(s) = s$ $\quad g(s) = 0$ $\quad h(s) = -\sqrt{1+s^2}$

to get $\phi(s)$, $\psi(s)$ we use

$$\begin{cases} F(x(s), y(s), u(s), \varphi(s), \psi(s)) = 0 \\ h'(s) = \frac{du}{dx} \cdot \frac{dx}{ds} + \frac{du}{dy} \cdot \frac{dy}{ds} \end{cases}$$

$$= \phi(s) f'(s) + \psi(s) g'(s)$$

$$\begin{cases} \varphi(s)^2 + \psi(s)^2 = \dfrac{1}{c^2} \\ -\dfrac{1}{2}\dfrac{2s}{\sqrt{1+s^2}} = \phi(s) \end{cases}$$

From this we get $\phi(s) = \dfrac{-s}{\sqrt{1+s^2}}$

$$\psi(s) = \pm\sqrt{\frac{1}{c^2} - \frac{s^2}{(1+s^2)}}$$

$F_x + F_u u_x +$

$\widehat{F_{u_x} u_{xx}}$

$F_{u_y} u_{yx}$

$\to$ We then solve for the characteristics

$$\frac{dx}{dt} = F_p \qquad \frac{dy}{dt} = F_q \qquad \frac{dz}{dt} = F_p p - F_q q$$

$$\frac{dp}{dt} = -F_x - F_u p \qquad \frac{dq}{dt} = -F_y - F_u q$$

$$\frac{du_x}{dt} = u_{xx}\overset{F_p}{\frac{dx}{dt}} + u_{xy}\frac{dy}{dt}$$

$$\Rightarrow F_p = 2u_x \qquad F_q = 2u_y$$

$$\frac{dx}{dt} = 2\underbrace{u_x}_{p} \qquad \frac{dy}{dt} = 2\underbrace{u_y}_{q} \qquad \frac{dz}{dt} = -2\underbrace{u_x \cdot u_x}_{p^2} - 2\underbrace{u_y u_y}_{q^2}$$

$$\frac{dp}{dt} = 0 \qquad \frac{dq}{dt} = 0$$

$$x(0) = s \qquad y(0) = 0 \qquad z(0) = -\sqrt{1+s^2}$$

$$p(0) = \frac{-s}{\sqrt{1+s^2}} \qquad q(0) = \pm\sqrt{\frac{1}{c^2} - \frac{s^2}{1+s^2}}$$

$$\Rightarrow \quad p(t,s) = \frac{-s}{\sqrt{1+s^2}} \qquad q(t,s) = \pm\sqrt{\frac{1}{c^2} - \frac{s^2}{1+s^2}}$$

$$u(t,s) = -2(p^2+q^2)t + \text{cst}$$

using $z(0)$

$$= -2(p^2+q^2)t + z(0)$$

$$= -2\left(\frac{s^2}{(1+s^2)} + \frac{1}{c^2} - \frac{s^2}{1+s^2}\right)t - \sqrt{1+s^2}$$

$$= -\frac{2t}{c^2} - \sqrt{1+s^2}$$

then $\dfrac{dx}{dt} = -\dfrac{2s}{\sqrt{1+s^2}}$ $\qquad$ $\dfrac{dy}{dt} = \pm 2\sqrt{\dfrac{1}{c^2} - \dfrac{s^2}{1+s^2}}$

using $x(0)$ $y(0)$

$$x(t,s) = -\frac{2s}{\sqrt{1+s^2}}\,t + s \qquad (**)$$

$$y(t,s) = \pm 2\sqrt{\frac{1}{c^2} - \frac{s^2}{1+s^2}}\;t \qquad (*)$$

using (*) we get

$$y^2 = 4\left(\frac{1}{c^2} - \frac{s^2}{1+s^2}\right)t^2$$

$$t = \pm\sqrt{\frac{y^2}{4\left(\frac{1}{c^2} - \frac{s^2}{1+s^2}\right)}}$$

(**) then gives

$$x\sqrt{1+s^2} = -2st + s\sqrt{1+s^2}$$

$$(x-s)\sqrt{1+s^2} = -2st$$

$$(x-s)^2(1+s^2) = 4s^2t^2 \rightarrow (x^2+s^2-2sx)(1+s^2) = 4s^2t^2$$

if we set $c=1$ then we have

$$p = \frac{-s}{\sqrt{1+s^2}} \qquad q = \pm\sqrt{1-\frac{s^2}{1+s^2}} = \pm\sqrt{\frac{1}{1+s^2}}$$

$$= \pm\frac{1}{\sqrt{1+s^2}}$$

$$y = \pm 2\sqrt{1-\frac{s^2}{1+s^2}}\, t = \pm 2 \frac{1}{\sqrt{1+s^2}}\, t$$

$$\Rightarrow t = \pm y \frac{\sqrt{1+s^2}}{2}$$

$$x = \frac{-2}{\sqrt{1+s^2}}\, t + s = \frac{-\cancel{2}}{\cancel{\sqrt{1+s^2}}}\left(\pm y \frac{\cancel{\sqrt{1+s^2}}}{\cancel{2}}\right) + s$$

$$x - s = \pm y \qquad s = x \pm y$$

Which we can substitute in $Z(s,t)$