Question 3

$$\begin{cases} \Delta u = \hat{f} & \text{on } B_R \\ u = 1 & \text{on } \partial B_R \end{cases}$$

2.1 How to find a solution?

2.2 in the case where $f = y$ → find explicit solution

Hint: use Fourier series for $F(r, \cdot)$ ←

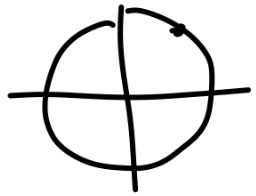
Solution: $\Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = f$

$u(r, \theta) = v(r) w(\theta)$

$$\Delta u = v''(r) w(\theta) + \frac{1}{r} v'(r) w(\theta) + \frac{1}{r^2} v(r) w''(\theta)$$

$\Delta u = 0 \to$ separate variables

$$\frac{v''(r) + \frac{1}{r} v'(r)}{\frac{1}{r^2} v(r)} = -\frac{w''(\theta)}{w(\theta)} = \lambda$$



$w''(\theta) = -\lambda w(\theta)$ $\qquad$ $w(\theta) = w(\theta + 2\pi)$

$w(\theta) = A \cos \mu\theta + B \sin \mu\theta$ $\leftarrow$ take $\lambda = \mu^2$

$$f(r,\theta) = a_0 + \sum_{k=1}^{\infty} A'_k \sin k\theta + B'_k \sin k\theta$$

$\sin \mu\theta = \sin(\mu(\theta + 2\pi))$

$\cos \mu\theta = \cos(\mu(\theta + 2\pi))$

$\to 2\pi\mu = 2\pi k \to \mu = k$

$u_k(r,\theta) = v_k(r)\, w_k(\theta)$

$$\sum_{k=1}^{\infty} v_k''(r)\left(A_k \sin k\theta + B_k \cos k\theta\right)$$

$$+ \sum_{k=1}^{\infty} v_k'(r)\frac{1}{r}\left(A_k \sin k\theta + B_k \cos k\theta\right)$$

$$- \frac{k^2}{r^2}\sum_{k=1}^{\infty}\left(A_k \sin k\theta + B_k \cos k\theta\right) v_k(r) = a_0 + \sum_{k=1}^{\infty} A_k'(r) \sin k\theta + \sum_{k=1}^{\infty} B_k'(r) \cos k\theta$$

assuming $a_0 = 0$

$$\begin{cases} v_k''(r) A_k + v_k'(r)\frac{1}{r} A_k - \frac{k^2}{r^2} A_k v_k(r) = A_k' \\ v_k''(r) B_k + v_k'(r)\frac{1}{r} B_k - \frac{k^2}{r^2} B_k v_k(r) = B_k' \end{cases}$$

$\rightarrow$ With $f = y$ $\quad f(r,\theta) = r\sin\theta$

$$\Rightarrow B_k' = 0 \;\forall k \qquad A_k' = 0 \;\forall k \neq 1$$

$$A_1' = r$$

Substituting in our ODE we get

$$v_1''(r) + v_1'(r)\frac{1}{r} - \frac{1}{r^2}v_1(r) = r$$

$r = e^s \quad w(s) \Rightarrow \quad w(\log(r)) = v_1(r)$

$\rightarrow \quad \rightarrow \quad v_1'(r) = w'(s)\cdot\frac{1}{r}$

$$v_1''(r) = w''(s)\left(\frac{1}{r}\right)^2 + w'(s)\left(-\frac{1}{r^2}\right)$$

$$w''(s)\frac{1}{r^2} - \frac{1}{r^2}w(s) = r$$

$$w''(s) - w(s) = r^3 = e^{3s}$$

$\rightarrow$ homogeneous: $\quad w''(s) - w(s) = 0 \rightarrow w(s) = e^s + e^{-s}$

particular solution : $\quad Ce^{3s} \rightarrow C9e^{3s} - Ce^{3s} = e^{3s}$

$C = 1/8$

→ Solution of ODE given by

$$W(s) = Ae^{s} + Be^{-s} + \frac{1}{8}e^{3s}$$

$$w_1(r) = Ar + \underbrace{Br^{-1}} + \frac{1}{8}r^3$$

to avoid blow up at 0 → $B = 0$

→ $u(r,\theta) = \left(Ar + \frac{1}{8}r^3\right)\left(\sin\theta\right)$

→ BC → $u(R,\theta) = 1$

let us consider a solution $u = \tilde{u} + 1$ where $\tilde{u}$

both $u, \tilde{u}$ satisfy $\Delta u = f$

and we can use $\tilde{u}$ to match the BCs

$$\tilde{u}(R,\theta)=0 \longrightarrow \left(AR+\frac{1}{8}R^3\right)=0$$

$$\longrightarrow A=-\frac{1}{8}R^2$$

Substituting this in our solution we get

$$u(r,\theta)=\frac{1}{8}\left(r^3-R^2r\right)\sin\theta+1$$

Exercise 6

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

step 2 $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$f(z) = \sum_{n=0}^{\infty} a_n (r \operatorname{cis} \theta)^n$$

$$= \sum_{n=0}^{\infty} a_n r^n \operatorname{cis} n\theta$$

$$\text{real}\,\{f(z)) = \sum_{n=0}^{\infty} a_n r^n \cos n\theta = u(r, \theta)$$

$$\text{imag}\,\{f(z)\} = \sum_{n=0}^{\infty} a_n r^n \sin n\theta = v(r, \theta)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{-y/x^2}{1+\frac{y^2}{x^2}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\,\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$\frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} a_n\, n\, r^{n-1} \cos n\theta \cdot \frac{x}{r} - \sum_{n=0}^{\infty} a_n r^n n \sin n\theta$$

$$\times \left(\frac{\partial}{\partial x} \operatorname{alg}\left(\frac{y}{x}\right)\right)$$

$$= \sum_{n=0}^{\infty} \overbrace{a_n\, n\, r^{n-2} \cos n\theta \cdot x}$$

$$- \sum_{n=0}^{\infty} a_n r^n \cdot n \sin n\theta \cdot \left(\frac{-y/x^2}{1+y^2/x^2}\right) \to \frac{-y}{\underbrace{x^2+y^2}_{r^2}}$$

On the other hand, we have

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$= \sum_{n=0}^{\infty} a_n r^{n-1} n \sin n\theta \cdot \frac{x}{r}$$

$$+ \sum_{n=0}^{\infty} a_n r^{n} n \cos n\theta \cdot \frac{1/x}{1+\frac{y^2}{x^2}}$$

$$= \sum_{n=0}^{\infty} a_n r^{n-2} n \sin n\theta \; x$$

$$+ \sum_{n=0}^{\infty} a_n r^{n} n \cos n\theta \; \frac{x}{\underbrace{x^2+y^2}_{r^2}}$$

Question 6 (cont^{d})

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + (i)^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (i)^2 \frac{\partial^2 v}{\partial y^2}$$

$$= \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 u}{\partial y \partial x} = 0$$

u harmonic $\Rightarrow u \in C^\infty$ ($\rightarrow$ see note on website)

(Schwarz theorem for derivative swap)

$$\rightarrow \sum_{i=1}^{\infty} \frac{\partial^2 u}{\partial x_i^2} = 0$$

$$\rightarrow \frac{\partial}{\partial x_j} \sum_{i=1}^{\infty} \frac{\partial^2 u}{\partial x_i^2} = 0$$

$$\sum_{i=1}^{\infty} \frac{\partial^2}{\partial x_i^2} \frac{\partial u}{\partial x_j} = 0 \rightarrow \Delta \frac{\partial u}{\partial x_j} = 0$$

$$u \in C^\infty \Rightarrow \frac{\partial u}{\partial x_j} \in C^2$$

$\rightarrow \frac{\partial u}{\partial x_j}$ harmonic

Question 2

Question 4

Schwarz inequality : $|\langle f, g \rangle| \leq \|f\| \, \|g\|$

$$\|f\| = \sqrt{\int |f|^2 dx} \qquad \langle f, g \rangle = \int f \cdot g \, dx$$

$$u(x) = \frac{1}{\mathrm{vol}(B_R(x))} \int_{B_R(x)} u(y) \, dy$$

Solution $\int u^2 dx < \infty \Rightarrow u(x) \equiv 0 \quad \forall x$

using C-S $\rightarrow$ $\displaystyle\int_{B_R(x)} u^2 dy \int_{B_R(x)} 1 \, dy \geq \left| \int_{B_R(x)} u \cdot 1 \, dy \right|^2$

On the other hand, using the mean value formula, we get

$$\int_{B_R(x)} u(y)\,dy = \mathrm{vol}(B_R(x))\,u(x)$$

$$\left(\mathrm{vol}(B_R(x))\,u(x)\right)^2 \leq \int_{B_R(x)} u^2(y)\,dy < +\infty$$

taking the limit as $R \to \infty$ we get

$$\lim_{R\to\infty} \mathrm{vol}(B_R(X))\; u(X) = \lim_{R\to\infty} \int_{B_R(x)} u^2(y)\,dy$$

$$= \int_{\mathbb{R}^n} u^2(y)\,dy < \infty$$

$$\Rightarrow \quad u(x) = 0$$

Question 9

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad u(a,\theta) = h(\theta)$$

$$\rightarrow \quad u(r,\theta) = v(r)\,w(\theta)$$

$$\left(v''(r) + \frac{1}{r}v'(r)\right)w(\theta) + \frac{1}{r^2}v(r)\,w''(\theta) \qquad W(\theta) = A\theta + B$$

$$\frac{r^2 v''(r) + r v'(r)}{v(r)} = \lambda = m^2 \qquad \frac{w''(\theta)}{w(\theta)} = -\lambda \rightarrow$$

$$v(r) = A r^{m} + \cancel{B r^{-m}}$$

not possible if we take
u bounded at $r=0$

$$W(\theta) = W(\theta + 2\pi) \to W_k(\theta) = A_k \cos k 2\pi \theta$$

$$\widehat{u(r,\theta)} = \sum_{m=1}^{\infty} A_m \cos m\theta + B_m \sin m\theta + C$$

$$h(\theta) = a_0 + \sum_{m=1}^{\infty} \alpha_m \cos m\theta + \beta_m \sin m\theta$$

$$\alpha_m = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos m\theta \, d\theta = A_m$$

$$\beta_m = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \sin m\theta \, d\theta = B_m$$

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_0^{2\pi} h(\theta) \sin n\theta' \, d\theta' \right) \sin n\theta$$

$$+ \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos n\theta' \, d\theta' \right) \cos n\theta$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{1}{\pi} \int_0^{2\pi} h(\theta) \left(\sin n\theta' \sin n\theta + \cos n\theta' \cos n\theta \right) d\theta'$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{1}{\pi} \int_0^{2\pi} h(\theta') \cos(n(\theta' - \theta)) \, d\theta'$$

We then $a_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \longrightarrow$

$$u(r,\theta) = \frac{1}{2} \quad \frac{1}{\pi}\int_0^{2\pi} h(\theta')\,d\theta' + \sum_{n=1}^{\infty} \frac{1}{\pi}\int_0^{2\pi} h(\theta')\cos\left(n(\theta'-\theta)\right)d\theta'$$

$$\frac{1}{\pi}\sum_{n=0}^{\infty}\int_0^{2\pi} \underbrace{h(\theta')\;\cos(n(\theta'-\theta))}\,d\theta' - 1/2$$

i)