$$\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3 \qquad \vec{u}(x_1, x_2, x_3)$$

$$\nabla \times u = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} = \hat{i}\,(\partial_{x_2} u_3 - \partial_{x_3} u_2) - \hat{j}\,(\partial_{x_1} u_3 - \partial_{x_3} u_1) + \hat{k}\,(\partial_{x_1} u_2 - \partial_{x_2} u_1)$$

$$\nabla(\nabla \cdot u) = \left[\frac{\partial}{\partial x_1}\left(\sum_{i=1}^{3} \frac{\partial u}{\partial x_i} \right), \frac{\partial}{\partial x_2}\left(\sum_{i=1}^{3} \frac{\partial u}{\partial x_i} \right), \frac{\partial}{\partial x_3}\left(\sum_{i=1}^{3} \frac{\partial u}{\partial x_i} \right) \right]$$

$$\nabla^2 u = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} u_1 + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \\ \frac{\partial^2}{\partial x_1^2} u_2 + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} \\ \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2} \end{bmatrix} \longleftarrow$$

$$\nabla \times (\nabla \times u) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ \partial_2 u_3 - \partial_3 u_2 & \partial_3 u_1 - \partial_1 u_3 & \partial_1 u_2 - \partial_2 u_1 \end{vmatrix}$$

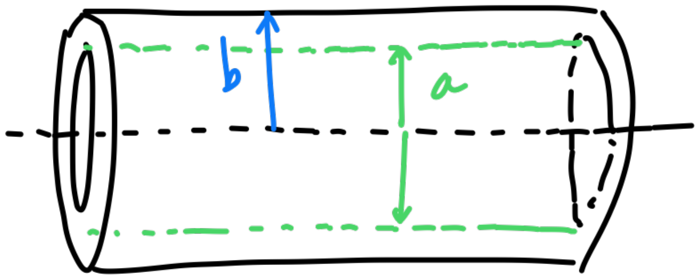
LHS

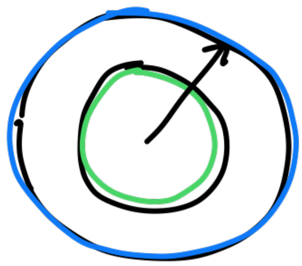
$$= \begin{bmatrix} \partial_2 \partial_1 u_2 - \partial_2^2 u_1 - \partial_3^2 u_1 + \partial_3 \partial_1 u_3 \\ -\partial_1^2 u_2 + \partial_1 \partial_2 u_1 + \partial_3 \partial_2 u_3 - \partial_3^2 u_2 \\ \partial_1 \partial_3 u_1 - \partial_1^2 u_3 - \partial_2^2 u_3 + \partial_2 \partial_3 u_2 \end{bmatrix}$$

$$
\begin{bmatrix}
\cancel{\dfrac{\partial^2 u_1}{\partial x_1^2}} + \dfrac{\partial^2 u_2}{\partial x_1 \partial x_2} + \dfrac{\partial^2 u_3}{\partial x_1 \partial x_3} \\
\dfrac{\partial^2 u_1}{\partial x_2 \partial x_1} + \cancel{\dfrac{\partial^2 u_2}{\partial x_2^2}} + \dfrac{\partial^2 u_3}{\partial x_2 \partial x_3} \\
\dfrac{\partial^2 u_1}{\partial x_3 x_1} + \dfrac{\partial^2 u_2}{\partial x_3 \partial x_2} + \cancel{\dfrac{\partial^2 u_3}{\partial x_3^2}}
\end{bmatrix}
-
\begin{bmatrix}
\cancel{\dfrac{\partial^2 u_1}{\partial x_1^2}} + \dfrac{\partial^2 u_1}{\partial x_2^2} + \dfrac{\partial^2 u_1}{\partial x_3^2} \\
\dfrac{\partial^2 u_2}{\partial x_1^2} + \cancel{\dfrac{\partial^2 u_2}{\partial x_2^2}} + \dfrac{\partial^2 u_2}{\partial x_3^2} \\
\dfrac{\partial^2 u_3}{\partial x_1^2} + \dfrac{\partial^2 u_3}{\partial x_2^2} + \cancel{\dfrac{\partial^2 u_3}{\partial x_3^2}}
\end{bmatrix}
$$

RHS

Question 3





$$\frac{\partial u}{\partial t} = \frac{k_0}{\rho c}\Delta u = \frac{k_0}{\rho c}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \underbrace{\frac{\partial^2 u}{\partial x_3^2}}\right)$$

$$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \\ x_3 = z \end{cases}$$

$$u(x_1, x_2, x_3) = u(r\cos\theta, r\sin\theta, z) = u(r, \theta, z)$$

$$\frac{\partial^2 u}{\partial x_1^2} \Rightarrow \qquad \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x_1} +$$

$$r = \sqrt{x_1^2 + x_2^2}$$

$$\frac{\partial r}{\partial x_1} = \frac{1}{2}\,\frac{2x_1}{(x_1^2 + x_2^2)^{1/2}}$$

$$\frac{\partial r}{\partial x_2} = \frac{1}{2}\,\frac{2x_2}{(x_1^2 + x_2^2)^{1/2}}$$

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x_1} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x_1}$$

$$\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x_2} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x_2}$$

$$\frac{\partial u}{\partial r}\cdot\cos\theta + \frac{\partial u}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial u}{\partial r}\cdot\sin\theta + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right)$$

$$\theta = \operatorname{atg}\left(\frac{x_2}{x_1}\right)$$

$$\frac{\partial \theta}{\partial x_1} = \frac{1}{1 + \left(\frac{x_2}{x_1}\right)^2}\cdot\frac{-x_2}{x_1^2} = \frac{\cancel{x_1^2}}{x_1^2 + x_2^2}\cdot\frac{-x_2}{\cancel{x_1^2}} = \frac{-x_2}{x_1^2 + x_2^2}$$

$$\frac{\partial \theta}{\partial x_2} = \frac{1}{1 + \left(\frac{x_2}{x_1}\right)^2}\cdot\frac{1}{x_1} = \frac{x_1^2}{x_1^2 + x_2^2}\cdot\frac{1}{x_1} = \frac{x_1}{x_1^2 + x_2^2}$$

Compute second order derivatives for the laplacian

$$\frac{\partial^2 u}{\partial x_1^2} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$\frac{\partial^2 u}{\partial x_2^2} = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}$$

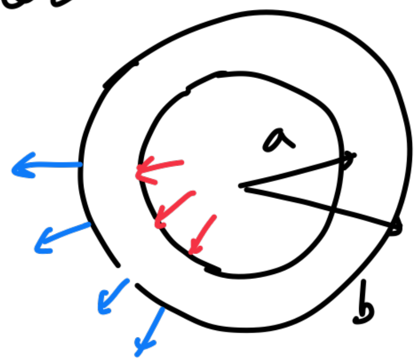
$$\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r}$$

$$+ \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

$$\frac{\partial u}{\partial t} = \frac{k_0}{\rho c}\Delta u = \frac{k_0}{\rho c}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right)$$

Or derive the laplacian from conservation of energy



$$\frac{\partial e(r,t)}{\partial t} 2\pi r\, dr$$

$$= \phi(r,t)\,2\pi r - \phi(r+dr,t)\,2\pi(r+dr)$$

$e(r,t) = c\rho u(r,t)$

$$c\rho \frac{\partial u}{\partial t} 2\pi r = \frac{1}{dr}\left[\phi(r,t)2\pi r - \phi(r+dr,t)\,2\pi \times (r+dr)\right]$$

$$c\rho \frac{\partial u}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r}\left[r\, \phi(r,t) \right]$$

$\downarrow$ use Fourier's law $\quad \phi(r,t) = -k_0 \dfrac{\partial u}{\partial r}$

$$c\rho \cancel{\frac{\partial u}{\partial t}} = \frac{1}{r} \frac{\partial}{\partial r}\left[r \frac{\partial u}{\partial r} \right]$$

0

<u>steady state equation</u> $\quad \cancel{\frac{1}{r}} \underbrace{\frac{\partial}{\partial r}\left[r \frac{\partial u}{\partial r} \right]} = 0$

$$\frac{\partial}{\partial r}\left[r \frac{\partial u}{\partial r} \right] = 0 \Rightarrow r \frac{\partial u}{\partial r} = \text{constant} = A$$

$$\frac{\partial u}{\partial r} = \frac{A}{r} \Rightarrow u = A \log r + B$$

$$u_{ss}(r) = A \log r + B$$

$$\left.\begin{array}{l} u(r=a) = T_a \\ u(r=b) = T_b \end{array}\right\} \longrightarrow \left\{\begin{array}{l} A \log a + B = T_a \\ A \log b + B = T_b \end{array}\right.$$

$A = \ldots.$

$B = \ldots.$