MATH-UA 9263 - Partial Differential Equations
PSet 3: Method of characteristics, first order equations

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Given date: April 6
Due date: April 24
Total: 20pts

Question 1 (5pts) *Consider the boundary value problem*

$$\begin{cases} u_x + 3xu_y = 0 & x \in \mathbb{R}, y \in \mathbb{R} \\ u|_{y=0} = f(x) & x \in \mathbb{R} \end{cases}$$

a) What condition(s) does f need to satisfy in order for a solution to exist?

b) In which region of the xy-plane is the solution uniquely determined by the initial condition?

Question 2 (Burgers equation (Part I), 5pts) *In this question, we consider the* inviscid Burgers equation

$$u_t + uu_x = 0$$

1. *Using the inverse function theorem and the value of the Jacobian, start by finding the breaking time of the solution of this equation when the initial condition is given by the sinusoidal wave $u(x, 0) = \sin x$. Where does the breaking occur?*

2. *We now want to solve the Cauchy problem*

$$\begin{cases} u_t + uu_x = 0 \\ \\ u(x, 0) = g(x) \end{cases}$$

$$g(x) = \begin{cases} 1 & x \leq 0 \\ 1 - x & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

[Consider a shock wave.]

3. Finally we consider the modified Burgers equation

$$u_t + u^2 u_x = 0$$

a) Show that the function $w = u^2$ satisfies the usual Burgers equation

b) On the basis of this result, find the solution of the initial value problem for the modified Burgers equation

$$\begin{cases} u_t + u^2 u_x = 0 \\ u(x,0) = x \end{cases}$$

c) What is the domain of validity of this solution?

Question 3 (5pts) *Find a solution to the following initial value problems and sketch the characteristics (including shock waves and fan like characteristics when needed)*

$$\begin{cases} 2xy\, u_x + (x^2 + y^2)\, u_y = 0 \\ u(x,y) = \exp(x/(x-y)) \qquad \text{on } x + y = 1 \end{cases}$$

$$\begin{cases} u_t + xt\, u_x = t \quad x \in \mathbb{R}, t > 0 \\ u(x,0) = x^2 \qquad x \in \mathbb{R} \end{cases}$$

$$\begin{cases} u_t + \left(\frac{u(2-u)}{2}\right)_x = 0 \quad x \in \mathbb{R}, t \geq 0 \\ u(x,0) = g(x) \qquad x \in \mathbb{R} \end{cases}$$

For $g(x)$ defined as

$$\text{(i)} \quad g(x) = \begin{cases} 2 & \text{if } x < 1 \\ 1 & \text{if } x \in]1,2[\\ 0 & \text{if } x > 2 \end{cases}$$

$$\text{(ii)} \quad g(x) = \begin{cases} 0 & x < 1 \\ 1 & x > 1 \end{cases}$$

Question 4 (Burgers equation (Part II), 5pts) *We again consider the Cauchy problem for the Burgers equation*

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Suppose that $\phi' \geq -C$ where $C > 0$. Show that the PDE has a C^1 solution on $(x,t) \in \mathbb{R} \times [0, \frac{1}{C})$. Then show that for $t \in [0, \frac{1}{C})$, u_x satifies the estimate

$$u_x(x,t) \geq \frac{1}{t - C^{-1}}$$

[Note that the right-hand side is negative. Hint: consider the difference quotients $\frac{u(\xi_2(t),t)-u(\xi_1(t),t)}{\xi_2(t)-\xi_1(t)}$ *where* $x = \xi_j(t)$ *are the (projected) characteristic curves emanating from point* x_j *on the* x*-axis]*

References

[1] Marcelo Epstein, *Partial Differential Equations Mathematical Techniques for Engineers*, Springer, 2017.

[2] Richard Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Fourth Edition, Pearson 2004.

[3] Sandro Salsa, *Partial Differential Equations in Action, From Modelling to Theory*, Springer, 2016.

[4] Adrás Vasy, *Partial Differential Equations, An Accessible Route through Theory and Applications*, Graduate studies in Mathematics Vol 169, AMS, 2015.