MATH-UA 9263. Partial Differential Equations
Material for the Midterm

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Theory

For the theory part, you must be able to:

1. Characterize a PDE in terms of its order and non linearity
2. Derive the heat equation from physical laws
3. Solve the one dimensional heat equation on simple domains (spheres, cylinders, half-disks, ...) with boundary and initial conditions, using the method of separation of variables and the Fourier series.
4. Understand and describe the various notions of convergence (L^2, pointwise and uniform) for the Fourier series
5. Understand and use the energy method to prove uniqueness of the solution for the heat equation in simple frameworks
6. State and prove the Weak Maximum principle for the heat equation
7. Understand the notion of fundamental solution for the heat equation. In particular, understand and explain how the fundamental solution can be derived from invariant transformations and give the expression of this solution.
8. Understand the notion of Cauchy problem for the heat equation. In particular provide the solution of the Cauchy problem for the one dimensional heat equation by relying on the Fundamental solution
9. Prove the existence of a solution for the Cauchy problem associated to the heat equation and prove that this solution obeys an exponential bound and satisfies the Cauchy data when this data satisfies an exponential growth condition of the form $|g(x)| \leq ce^{ax^2}$.
10. Solve the non homogeneous Cauchy problem for the heat equation with zero and non zero initial data by means of the Fundamental solution and of Duhamel's method
11. Prove uniqueness of the solution of the Cauchy problem when the solutions are restricted to functions that obey an exponential growth condition of the form $|u(x,t)| \leq Ae^{a|x|^2}$
12. Define and explain the notion of harmonic function
13. State and prove the mean value formulas for harmonic functions

14. State and prove Harnack's inequality for harmonic functions
15. Derive the fundamental solution in dimension 2 and 3 for Laplace's equation, in particular explain how this solution can be derived from the invariants of Laplace's equation and provide the general expression of this solution in dimension n
16. Understand and Explain the notion of Newtonian potential. Prove that this potential gives the unique solution of Poisson's equation, when considering twice continuously differentiable solutions that are vanishing at infinity and a source term that is twice continuously differentiable and has compact support.
17. State and use the Divergence Theorem and integration by parts and derive Green's formulas from the Divergence theorem and integration by parts.
18. Derive the integral formulation for a function $u \in C^2(\overline{U})$ and the fundamental solution Φ of Laplace's equation.
19. Understand the notion of Green's function and its importance for the integral formulation.
20. Provide the general definition of the Green function and compute the Green function for simple geometries (e.g. half space, disk, strip ...) using the method of images (see corresponding exercises in the recitations below).

Recitations

You must be able to solve all the exercises that were solved during the recitations. This includes

1. **Recitation 1**: All the questions
2. **Recitation 2**: Question 1, 2, 6, 7, 10, 13
3. **Recitation 3**: Questions 8, 14,15
4. **Recitation 4**: Questions 2, 3, 4, 6, 9
5. **Recitation 5**: Questions 1, 3, 6, 7, 4