Partial Differential Equations, lecture 3

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Heat equation and separation of variables

We want to study the temperature evolution during an interval of time, say from $t = 0$ to $t = T$.

As different initial configurations will lead to different evolutions of temperature, it seems reasonable to prescribe the initial temperature profile $u(x, 0) = g(x)$.

Specifying initial conditions is however not enough to determine a unique evolution. It is also necessary to know how the bar interacts with its surroundings. Starting with a given intial temperature distribution, we can indeed change the evolution of u by controlling the temperature or the heat flux at the two ends of the bar (recall that we assumed the bar had perfect lateral insulation). We could for example keep the temperature at a certain fixed level or let it vary in a certain way depending on time. This amounts to prescribing conditions of the form

$$u(0, t) = h_1(t), \quad u(L, t) = h_2(t) \tag{1}$$

at any time $t \in (0, T]$. Such conditions are known as Dirichlet boundary conditions.

We could also prescribe the heat flux at the end points. Starting from Fourier's law, we get

$$\text{(Inward heat flow at } x = 0) \quad -k_0 u_x(0, t) \tag{2}$$

$$\text{(Inward heat flow at } x = L) \quad k_0 u_x(L, t) \tag{3}$$

The heat flux is assigned through the Neumann boundary conditions,

$$-u_x(0,t) = h_1(t), \quad u_x(L,t) = h_2(t) \tag{4}$$

at any time $t \in (0,T]$.

A last common type of boundary condition is the Robin or radiation condition. Let the surroundings of the bar be kept at temperature U and assume that the inward heat flux from one end of the bar depends linearly on the difference $U - u$ (such an assumption can be related to Newton's law of cooling according to which the heat loss from the surface of a body is a linear function of the temperature drop $U - u$ from the surroundings to the surface). We then have

$$k_0 u_x = \gamma(U - u), \quad (\gamma > 0) \tag{5}$$

Letting $\alpha = \gamma/k_0 > 0$ and $h = \gamma U/k_0$. The Robin conditions at $x = L$ can be summarized as

$$u_x + \alpha u = h \tag{6}$$

Clearly, it is also possible to assign mixed conditions (for instance a Dirichlet condition at one end and a Neumann condition at the other).

The problems associated with the above boundary conditions have a corresponding nomenclature.

Summarizing, we can state the most common problems for the one dimensional heat equation as follows.

Given $f = f(x,t)$ (external source) and $g = g(x)$ (intial or Cauchy data), determine $u = u(x,t)$ such that

$$\begin{cases} u_t - Du_{xx} = f & 0 < x < L,\ 0 < t < T \\ u(x,0) = g(x) & 0 \le x \le L \\ + \text{ boundary conditions} & 0 \le t \le T \end{cases} \tag{7}$$

where the boundary conditions include

- Dirichlet : $u(0,t) = h_1(t)$, $u(L,t) = h_2(t)$
- Neumann $-u_x(0,t) = h_1(t)$, $u_x(L,t) = h_2(t)$

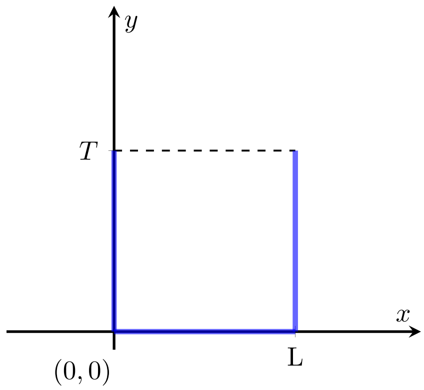

Figure 1: Parabolic boundary

- Robin or radiation

$$-u_x(0,t) + \alpha u(0,t) = h_1(t) \qquad u_x(L,t) + \alpha u(L,t) = h_2(t)$$

Or any combination. Accordingly we have the initial (or Cauchy) Dirichlet problem, the initial Neumann problem and so on. When $h_1 = h_2 = 0$, we say that the boundary conditions are homogeneous.

Observe that only a special part of the boundary of the rectangle $Q_T = (0, L) \times (0, T)$ (shown in Fig. 1) called the parabolic boundary of Q_T carries the data. No Final condition (for $t = T$, $0 < x < L$) is required.

We will now prove that under reasonable hypotheses, the initial Dirichlet, Neumann or Robin and mixed problems are well posed. Sometimes this can be shown using elementary techniques like the method of separation of variables.

A before, we consider our one dimensional bar of length L. We assume that this bar is initially (at time $t = 0$) kept at constant temperature u_0. After that, the end point $x = 0$ is kept at the same temperature while the other end, at $x = L$ is kept at a constant temperature $u_1 > u_0$. We want to know how the temperature evolves inside the bar.

Before making any computation, let us try to conjecture what could happen.

- Given that $u_1 > u_0$, heat should start flowing from the hotter end, raising the temperature inside the bar and causing a heat outflow inside the cold boundary
- On the other hand, the interior increase of temperature should cause the hot inflow to decrease in time while the outflow increases
- We thus expect that sooner or later the two fluxes will balance each other so that the temperature eventually reaches a steady state distribution (it would also be interesting to know how fast the steady state is reached)

We will see that this is exactly the behavior predicted by our mathematical model given by the heat equation

$$u_t - Du_{xx} = 0, \qquad t > 0, 0 < x < L \tag{8}$$

with the initial Dirichlet conditions

$$\begin{cases} u(x,0) = u_0 & 0 \leq x \leq L \\ u(0,t) = u_0 & t > 0 \\ u(L,t) = u_1 & t > 0 \end{cases} \tag{9}$$

To start we introduce dimensionless variables, that is variables independent of the units of measurement. To do that, we rescale space, time and temperature with respect to quantities that are characteristic of our problem.

- For the space variable, we can use the length L of the bar as a rescaling factor, setting

$$y = \frac{x}{L} \tag{10}$$

  which is clearly dimensionless, being a ratio of lengths. Notice that $0 \leq y \leq 1$.

- How can we rescale time? Observe that the physical dimensions of the diffusion coefficient D are $[\text{length}]^2 \times [\text{time}]^{-1}$. Thus the constant $\tau = \frac{L^2}{D}$ gives a characteristic time scale for our diffusion problem. From this we can thus introduce the dimensionless time

$$s = \frac{t}{\tau} \tag{11}$$

  Finally we rescale the temperature by setting

$$z(y,s) = \frac{u(Ly, \tau s) - u_0}{u_1 - u_0} \tag{12}$$

For the dimensionless temperature z, we now get

$$z(y,0) = \frac{u(Ly,0) - u_0}{u_1 - u_0} = 0, \quad 0 \leq y \leq 1 \tag{13}$$

$$z(0,s) = \frac{u(0,\tau s) - u_0}{u_1 - u_0} = 0, \qquad z(1,s) = \frac{u(L,\tau s) - u_0}{u_1 - u_0} = 1 \tag{14}$$

Let us now rewrite the heat equation with these dimensionless quantities. Note that we have

$$u(x,t) = (u_1 - u_0) z\left(\frac{x}{L}, \frac{t}{\tau}\right) + u_0 \tag{15}$$

From this, using the chain rule, we get

$$\frac{\partial u}{\partial t} = (u_1 - u_0)\frac{1}{\tau}\frac{\partial z}{\partial s} \tag{16}$$

as well as

$$\frac{\partial u}{\partial x} = (u_1 - u_0)\frac{1}{L}\frac{\partial z}{\partial y} \tag{17}$$

and hence

$$\frac{\partial^2 u}{\partial x^2} = (u_1 - u_0)\frac{1}{L^2}\frac{\partial^2 z}{\partial y^2} \tag{18}$$

Grouping those two terms, and substituting into the heat equation, we get

$$(u_1 - u_0)\left(\frac{1}{\tau}\frac{\partial z}{\partial s} - \frac{D}{L^2}\frac{\partial^2 z}{\partial y^2}\right) = 0 \tag{19}$$

using $D = \frac{L^2}{\tau}$, this finally gives

$$z_s - z_{yy} = 0 \tag{20}$$

together with the initial conditions

$$z(y,0) = 0 \tag{21}$$

and the Dirichlet/Boundary conditions

$$z(0,s) = 0, \quad z(1,s) = 1 \tag{22}$$

We see that in the dimensionless formulation, the parameters L and D have disappeared, emphasizing the mathematical essence of the problem.

Steady state solution

We start by solving the Cauchy/Dirichlet problem by focusing on the steady state solution z^{st} that satisfies the equation $z_{yy} = 0$ and the boundary conditions (22).

An elementary computation gives

$$z^{st}(y) = Ay + B \tag{23}$$

which, with the BC's $z(0,s) = 0$, $z(1,s) = 1$ immediately gives $B = 0$ and $A = 1$. Hence we see that the steady state solution can be defined as $z^{st} = y$.

Going back to the original variables, we thus get

$$u^{st}(x,t) = (u_1 - u_0)z^{st}\left(\frac{x}{L}, \frac{t}{\tau}\right) + u_0 \tag{24}$$

$$= (u_1 - u_0)\frac{x}{L} + u_0 \tag{25}$$

Corresponding to a uniform heat flux along the bar given by Fourier's law

$$\varphi = -k_0 u_x = -k_0 \frac{u_1 - u_0}{L} \tag{26}$$

Transient regime

Knowing the steady state solution, it is convenient to introduce the function

$$U(y,s) = z^{st}(y,s) - z(y,s) = y - z(y,s) \tag{27}$$

Since we expect our solution to eventually reach the steady state, U represents what we call a transient regime that should converge to 0 as $s \to \infty$

Furthermore, the rate of convergence to 0 of U gives information on how fast the temperature reaches its equilibrium distribution. U satisfies the dimensionless heat equation with the initial condition $U(y,0) = y$ (i.e. $z(y,0) = 0$) and the homogeneous boundary conditions

$$U(0,s) = 0, \quad \text{and} \quad U(1,s) = 0 \tag{28}$$

Separation of variables

We are now in a position to find an explicit formula for U using the method of separation of variables. The main idea is to exploit the linear nature of the problem, constructing the solution by superposition of simpler solutions of the form $w(s)v(y)$ in which the variables s and y appear in separated form.

It is important to keep in mind that the reduction to homogeneous boundary conditions is crucial to carry on the computations.

We start by looking for non trivial solutions in the Cauchy/Dirichlet problem of the form

$$U(y,s) = w(s)v(y) \tag{29}$$

with $v(0) = v(1) = 0$ given by the boundary conditions. Substituting this in our dimensionless heat equation, we get

$$0 = \frac{\partial U}{\partial s} - \frac{\partial^2 U}{\partial y^2} = w'(s)v(y) - w(s)v''(y) \tag{30}$$

From which we can separate the variables as

$$\frac{w'(s)}{w(s)} = \frac{v''(y)}{v(y)} \tag{31}$$

Now the left-hand side is a function of s only while the right-hand side is a function of y only and the equality must hold for every $s > 0$ and every $y \in (0, L)$. Such a relation is possible only when both sides are equal to a common constant λ. We can thus write

$$v''(y) = \lambda v(y) \tag{32}$$
$$w'(s) = \lambda w(s) \tag{33}$$

together with $v(0) = v(1) = 0$. Let us start with (33). We consider three frameworks

(a) $\lambda = 0$

(b) $\lambda = \mu^2 > 0$

(c) $\lambda = -\mu^2 < 0$

In the first case, if $\lambda = 0$, we get

$$v''(y) - \lambda v(y) = 0 \tag{34}$$

which gives $v(y) = Ay + B$. Using $v(0) = v(1) = 0$, we thus obtain $A = B = 0$

In the second case, if $\lambda = \mu^2 > 0$ we get $v''(y) = \lambda v(y)$ from which

$$\frac{v''(y)}{v(y)} = \lambda \quad \Rightarrow \quad v(y) = Ae^{\mu y} + Be^{-\mu y} \tag{35}$$

using $v(0) = v(1) = 0$, we get $A + B = 0$ as well as $Ae^{\mu} + Be^{-\mu}$ which together imply $A(e^{\mu} - e^{-\mu}) = 0$ and hence $A = B = 0$

Finally for $\lambda = -\mu^2 < 0$ we get $v(y) = A^{i\mu y} + Be^{-i\mu y}$. Taking $v(0) = v(1) = 0$, we obtain

$$\begin{cases} A + B = 0 \\ Ae^{i\mu} + Be^{-i\mu} = 0 \end{cases}$$

which together imply $A(e^{i\mu} - e^{-i\mu}) = 0$ and hence $2 \sin \mu = 0$. In this last setting, for U to satisfy the boundary conditions, the constant μ must therefore be of the form $\mu = k\pi$, $k = 1, 2, \ldots$. Note that $k = 0$ gives the trivial solution $\lambda = 0$.

All in all, we can thus write

$$v_k(y) = A \sin k\pi y, \quad k = 1, 2, \ldots \tag{36}$$

With $\lambda = -\mu_k^2 = -(k\pi)^2$, our second eigenvalue problem turns to

$$\frac{w'(s)}{w(s)} = -(k\pi)^2 \tag{37}$$

with corresponding solution

$$w_k(s) = Ae^{-(k\pi)^2 s}, \quad k = 1, 2, \ldots \tag{38}$$

grouping (36) and (38) we get

$$U_k(y,s) = Ae^{-(k\pi)^2 s}\sin k\pi y, \quad k = 1, 2, \dots \tag{39}$$

Although our solution satisfies the homogeneous Dirichlet conditions, they do not match the initial condition $U(y,0) = y$ yet.

As we are solving a homogeneous problem

$$\begin{cases} U_s - U_{yy} = 0 \\ U(y,0) = y \\ U(0,s) = U(1,s) = 0 \end{cases}, \tag{40}$$

we can then try to construct the correct solution by superposing the U_k, that is to say, by considering a general solution of the form

$$U(y,s) = \sum_{k=1}^{\infty} A_k e^{-k^2\pi^2 s}\sin k\pi y. \tag{41}$$

From this, our initial conditions then require

$$U(y,0) = \sum_{k=1}^{\infty} A_k \sin k\pi y = y, \quad 0 \leq y \leq 1 \tag{42}$$

A number of questions then arise:

1. Is it possible to choose the coefficients A_k in order to satisfy this initial condition?
2. Since our solution involves an infinite expansion, we should also clarify in which sense U attains those initial conditions (as an example, do we have $U(z,s) \to y$ when $(z,s) \to (y,0)$?)
3. Any finite linear combination o the U_k is a solution of the heat equation, but how can we make sure that the same is true for U (i.e. an infinite combination of terms?). The answer to this question is positive if we can differentiate term by term the infinite sum so that

$$(\partial_s - \partial_{yy})U(y,s) = \sum_{k=1}^{\infty} (\partial_s - \partial_{yy})U_k(y,s) = 0 \tag{43}$$

4. Last but not least, even if we provide a positive answer to questions 1 and 2 above, when can we guarantee that U is the unique solution to our problem and therefore, that it describes the correct evolution of temperature inside the rod?

Our first question is rather general (we will see that it shows up again in the treatment of the wave equation) and involves the notion of Fourier series (which we address in the next section) of the initial data $f(y) = y$ on the interval $(0, 1)$.

Fourier series

When dealing with the heat equation, and in particular, with the method of separation of variables, we discovered that our initial conditions could be satisfied only if $f(x)$ could be equated to an infinite linear combination of eigenfunctions of a given boundary value problem. We will begin by investigating series of both sines and cosines (expansions in sines and cosines can then be derived as special cases). Let u be a $2T$-periodic function in $\mathbb{R}$ and assume that u can be expanded in a trigonometric series as follows

$$u(x) = U + \sum_{k=1}^{\infty} \left\{ a_k \cos \frac{\pi k}{T} x + b_k \sin \frac{\pi k}{T} x \right\} \tag{44}$$

The first question we should ask ourselves is What is the expression of the coefficients a_k, b_k and U?

To answer this question, we will use the orhogonality of the trigonometric functions (the proof is left as an exercise). Let $\omega = \frac{\pi}{T}$. Note that for a function that is $2T$ periodic, the sines and cosines are of the form $\sin(k\pi x/T)$ and $\cos(k\pi x/T)$.

$$\int_{-T}^{T} \cos(k\omega x) \cos m\omega x = \int_{-T}^{T} \sin k\omega x \sin m\omega x \, dx = 0, \quad \text{if } k \neq m \tag{45}$$

$$\int_{-T}^{T} \cos k\omega x \sin m\omega x \, dx = 0, \quad \text{for all } k, m \geq 0 \tag{46}$$

and finally

$$\int_{-T}^{T} \cos^2 k\omega x \, dx = \int_{-T}^{T} \sin^2 k\omega x \, dx = T \tag{47}$$

Now suppose that the series (44) converges uniformly in $\mathbb{R}$. Multiplying this series by $\cos n\omega x$ and integrating term by term over $(-T, T)$, the orthogonality relations (45), (46) and (47) give

$$\int_{-T}^{T} u(x) \cos n\omega x \, dx = \int_{-T}^{T} U \cos n\omega x \, dx + \sum_{k=1}^{\infty} a_k \int_{-T}^{T} \cos k\omega x \cos n\omega x \, dx \tag{48}$$

$$+ \sum_{k=1}^{\infty} b_k \int_{-T}^{T} \sin k\omega x \cos n\omega x \, dx \tag{49}$$

$$= \frac{U}{n\omega} \left| \sin(n\omega T) - \sin(-n\omega T) \right| + T a_n = T a_n \tag{50}$$

From which we therefore get

$$a_n = \frac{1}{T} \int_{-T}^{T} u(x) \cos n\omega x \, dx \tag{51}$$

and for $n = 0$, we have $2UT = \int_{-T}^{T} u(x) \, dx$. Then setting $U = \frac{a_0}{2}$ with

$$a_0 \equiv \frac{1}{T} \int_{-T}^{T} u(x) \, dx \tag{52}$$

Similarly we have for every n,

$$b_n = \frac{1}{T} \int_{-T}^{T} u(x) \sin n\omega x \, dx \tag{53}$$

From the above, we see that if $u(x)$ has a uniformly convergent expansion, the coefficients a_n, b_n must be given by formula (51) to (53). In this case, we say that the trigonometric series

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos k\omega x + b_k \sin k\omega x\} \tag{54}$$

is the Fourier series of u and the coefficients $a_0, a_1, \ldots, b_1, \ldots$ are the called the Fourier coefficients of u.

- if $u(x)$ is an odd function, i.e. $u(-x) = -u(x)$, then clearly we have

$$a_n = \frac{1}{T}\int_{-T}^{T} u(x)\cos n\omega x \, dx = 0 \tag{55}$$

  and the Fourier series reduces to a sine series

$$u(x) = \sum_{k=1}^{\infty} b_k \sin k\omega x \tag{56}$$

- If $u(x)$ is an even function, i.e. $u(x) = u(-x)$, then we have

$$b_n = \frac{1}{T}\int_{-T}^{T} u(x)\sin n\omega x \, dx = 0 \tag{57}$$

  and the Fourier series reduces to a cosine series

$$u(x) = \sum_{k=1}^{\infty} a_k \cos k\omega x + \frac{a_0}{2} \tag{58}$$

if we use Euler's identity $e^{ik\omega x} = \cos k\omega x + i \sin k\omega x$, the Fourier series (44) can be written as the sum

$$\sum_{k=-\infty}^{\infty} c_k e^{ik\omega x} \tag{59}$$

where the complex coefficients are defined as

$$c_k = \frac{1}{2T}\int_{-T}^{T} u(z)e^{-ik\omega z} \, dz \tag{60}$$

and the relation between the real and complex coefficients is given by

$$\begin{aligned} c_0 &= \frac{1}{2}a_0 \\ c_k &= \frac{1}{2}(a_k - b_k) \\ c_{-k} &= \overline{c_k} \end{aligned}$$

So far we have assumed that the function $u(x)$ admitted a Fourier series. A couple of questions thus remain:

1) Which conditions on u do ensure the "convergence" of its Fourier series? (and in what sense can we guarantee convergence of the series?)

2) If the Fourier series is convergent in some sense, does it always have sum u?

The convergence of the Fourier series is a delicate matter. We will only mention the most popular results (see the additional note on the course website for more details). We will successively cover the least squares (or L^2), the pointwise and the uniform convergence.

- Least squares or L^2 convergence. This is perhaps the most natural form of convergence for the Fourier series. Let S_N denote the N-partial sum of the Fourier series of u. We can then write

Theorem 1 *Let u be a square integrable function $\left(\int_{-T}^{T} u^2\, dx < \infty\right)$ on $(-T,T)$ then*

$$\lim_{N\to+\infty}\int_{-T}^{T}|S_N(u)-u(x)|^2\ dx = 0 \tag{61}$$

Moreover, the following Parseval relation holds

$$\frac{1}{T}\int_{-T}^{T}u^2 = \frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2) \tag{62}$$

Since the numerical series on the RHS of (62) is convergent, we deduce that $\lim_{k\to+\infty} a_k = \lim_{k\to+\infty} b_k = 0$. The result is sometimes known as the Riemann-Lebesgue lemma.

- We now discuss pointwise convergence. For this we first introduce the Dirichlet conditions. We say that a function $u(x)$ satisfies the Dirichlet conditions on $[-T,T]$ if it is continuous on $[-T,T]$ except possibly at a finite number of points of jump discontinuity and if the interval $[-T,T]$ can be partitioned in a finite number of subintervals such that u is monotonic in each of them. From this we can derive the following theorem which guarantees pointwise convergence

Theorem 2 *if u satisfies the Dirichlet conditions on $[-T,T]$ then the Fourier series of u converges at each point of $[-T,T]$. Moreover, setting $f(x\pm)=\lim_{y\to\pm x} f(y)$,*

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}\{a_k\cos k\omega x + b_k \sin k\omega x\} = \begin{cases} \frac{u(x+)+u(x-)}{2} & x\in(-T,T)\\ \frac{u(T-)+u(-T+)}{2} & x=\pm T\end{cases} \tag{63}$$

In particular, at every point x of continuity of u, the Fourier series converges to $u(x)$.

- Finally, uniform convergence can be related to the Weierstrass test. Since

$$|a_k\cos k\omega x + b_k\sin k\omega x| \le |a_k|+|b_k| \tag{64}$$

It follows that if the numerical series

$$\sum_{k=1}^{\infty} |a_k|, \qquad \sum_{k=1}^{\infty} |b_k| \tag{65}$$

are convergent, then the Fourier series of u is uniformly convergent in R with sum u. This is the case if $u \in \mathcal{C}^1(\mathbb{R})$ and is $2T$-periodic.

Conclusion and general remarks

Now that we have introduced the notion of Fourier series and given the form of our initial condition (42), it makes sense to wonder if we can write the RHS (Cauchy data) in this relation as a an expansion of sinusoids. In the case of problem (40), we thus look for a sine expansion of the 2 periodic and odd function that agrees with $f(y) = y$ on the $[-1, 1]$ interval. If

$$f(y) = \sum_{k=1}^{\infty} A_k \sin k\pi y, \tag{66}$$

the coefficients A_k are given by

$$A_k = 2\int_0^1 y \sin k\pi y \, dy = -\frac{2}{k\pi} \left| y \cos k\pi y \right|_0^1 + \frac{2}{k\pi} \int_0^1 \cos k\pi y \, dy \tag{67}$$

$$= -\frac{2\cos k\pi}{k\pi} = (-1)^{k+1} \frac{2}{k\pi} \tag{68}$$

The sine Fourier expansion of $f(y)$ on the interval $(0, 1)$ can therefore read as

$$y = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k\pi} \sin k\pi y \tag{69}$$

Clearly this expansion cannot be true at $y = 1$ since $\sin k\pi = 0$ for every k and this would therefore lead to the contradiction $1 = 0$. The theory of Fourier series implies that the expansion (69) is true at every point $y \in [0, 1)$.

In fact we can show that the relation (69) holds in the quadratic mean sense (or $L^2(0, 1)$ sense) that is

$$\int_0^1 \left[y - \sum_{k=1}^{N} (-1)^{k+1} \frac{2}{k\pi} \sin k\pi y \right]^2 dy \to 0, \quad \text{as } N \to \infty \tag{70}$$

From the definition of the A_k's we then obtain the final solution

$$U(y,s)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{k\pi}e^{-k^2\pi^2 s}\sin k\pi y \tag{71}$$

The solution (71) again satisfies the IC's in the least squares sense

$$\lim_{s\to 0^+}\int_0^1 \left[U(y,s)-y\right]^2\,dy=0. \tag{72}$$

In fact, by Parseval's identity we have

$$\int_0^1 \left[U(y,s)-y\right]^2\,dy=\frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{(e^{-k^2\pi^2 s}-1)^2}{k^2} \tag{73}$$

$$\leq \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{k^2} \tag{74}$$

since the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges, we can pass the limit $\lim_{s\to 0^+}$ inside the sum which gives

$$\lim_{s\to 0^+}\int_0^1 \left[U(y,s)-y\right]^2\,dy=0 \tag{75}$$

The analytical expression of U is rather reassuring. Our solution is a superposition of sinusoids of increasing frequencies k and of strongly damped amplitude because of the negative exponential.

On top of verifying the initial conditions, the particular nature of the solution also requires some care when discussing differentiability. The fact that each particular solution U_k is a valid solution (with respect to differentiability) by construction (as is any finite linear combination of those solutions), does not imply that an infinite sum of such solutions remains differentiable. This idea (which is known as term by term differentiation) is summarized by Theorem 3 below

Theorem 3 (Term by term differentiation) . *Suppose that $f_k : U \to \mathbb{R}$ for each $k = 1, 2, \ldots$ has continuous derivative on U (at the end point this means one-sided derivative). Suppose further that*

(i) The series $\sum_{k=1}^{\infty} f_k(x_0)$ converges at some point $x_0 \in U$ and

(ii) The series of derivatives $\sum_{k=1}^{\infty} f'_k(x)$ converges uniformly on U, to $f(x) = \sum_{k=1}^{\infty} f'_k(x)$

Then

1) The series $\sum_{k=1}^{\infty} f_k(x)$ converges at every $x \in U$ and the sum $F(x) = \sum_{k=1}^{\infty} u_k(x)$ is differentiable with $F'(x) = f(x)$ for each $x \in U$

2) Moreover, the convergence of $\sum_{k=1}^{\infty} f_k(x)$ to $F(x)$ is uniform on U.

In our case, the rapid convergence to zero of each term and its derivatives allows us to differentiate term by term. Indeed we have

$$\frac{\partial U_k}{\partial s} = \frac{\partial^2 U_k}{\partial y^2} = (-1)^k (2\pi k) e^{-k^2\pi^2 s} \sin k\pi y$$

so that if $s \geq s_0 > 0$ we get

$$\left|\frac{\partial U_k}{\partial s}\right|, \left|\frac{\partial^2 U_k}{\partial y^2}\right| \leq 2k\pi e^{-k^2\pi^2 s_0} \tag{76}$$

since the numerical series

$$\sum_{k=1}^{\infty} k e^{-k^2\pi^2 s_0}$$

converges[1] We can conclude by the Weierstrass test that the series

$$\sum_{k=1}^{\infty} \frac{\partial U_k}{\partial s}, \qquad \sum_{k=1}^{\infty} \frac{\partial^2 U_k}{\partial y^2}$$

converge uniformly in $[0,1] \times [s_0, \infty]$

we are left with checking the Dirichlet conditions

$$\begin{aligned} U(z,s) \to 0, &\quad \text{as} \quad (z,s) \to (0, s_0) \\ U(z,s) \to 0 &\quad \text{as} \quad (z,s) \to (1, s_0) \end{aligned}$$

This is true because we can take the limit under the sum due to the uniform convergence of the series

Uniqueness

We finally conclude with a discussion on the uniqueness of the solution (71). To show uniqueness, we will use the energy method. Suppose W is another solution of the Cauchy-Dirichlet problem (40). By linearity $v = U - W$ satisfies $v_s - v_{yy} = 0$ and has zero initial and boundary data.

Multiplying $v_s - v_{yy}$ and integrating in y over the interval $[0,1]$ we get

$$\int_0^1 v\, v_s\, dy - \int_0^1 v\, v_{yy}\, dy = 0$$

[1] if you need to convince yourself, check the ratio

$$\lim_{k\to\infty} \frac{(k+1)e^{-(k+1)^2 \pi 62 s_0}}{k e^{-k^2\pi^2 s_0}}$$

Observe that

$$\int_0^1 v\, v_s\, dy = \frac{1}{2}\int_0^1 \frac{\partial}{\partial s}\left(v^2\right)\, dy = \frac{1}{2}\frac{d}{ds}\int_0^1 v^2\, dy$$

Moreover,

$$\begin{aligned}\int_0^1 v\, v_{yy}\, dy = {} & [v(1,s)v_y(1,s) - v(0,s)v_y(0,s)] \\ & - \int_0^1 (v_y)^2\, dy\end{aligned}$$

Since $v(1,s) = v(0,s) = 0$, we get

$$\frac{1}{2}\frac{d}{ds}\int_0^1 v^2\, dy = -\int_0^1 (v_y)^2\, dy \leq 0 \tag{77}$$

which implies that the non negative function

$$E(s) = \frac{1}{2}\int_0^1 v^2\, dy$$

is non increasing. On the other hand, recall that any solution satisfies the initial data in the least squares sense so that both W and U satisfy

$$\lim_{s\to 0}\int_0^1 \left(U(s,y) - y\right)^2\, dy = 0$$
$$\lim_{s\to 0}\int_0^1 \left(W(s,y) - y\right)^2\, dy = 0$$

From those we get

$$\lim_{s\to 0}\int_0^1 \left(v(s,y)\right)^2\, dy = 0$$

which together with (77) finally implies $E(s) = 0$ for all values s and hence $v(s,y) = 0$ for all s.

References

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