First order equations   transport equation
Conservation laws

+ Method of characteristics

Part I

$$u_t + q(u)_x = 0 \qquad x \in \mathbb{R} \qquad t > 0 \qquad (*)$$

relation between density and flux $\rightarrow$ conservation law

u = density
concentration of physical quantity Q

$q(u)$ = flux function

evolution of Q in $[x_1, x_2]$

$$\int_{x_1}^{x_2} u(x,t)\,dx$$

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = -q(u(x_2,t)) + q(u(x_1,t))$$

if q regular

$$\int_{x_1}^{x_2} u_t(x,t)\,dx = \int_{x_1}^{x_2} q(u(x,t))_x\,dx$$

$$\int_{x_1}^{x_2} \left(u_t(x,t) - q(u(x,t))_x\right) dx \quad \forall\, x_1, x_2$$

$$\Rightarrow \quad u_t(x,t) - q(u(x,t))_x = 0$$

let us assume $\quad q = v u \qquad v$ scalar, constant

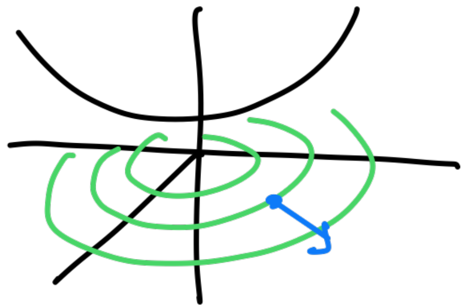
$v\hat{i} \rightarrow$ advection velocity

(linear convection setting)

$\longrightarrow$ (*) occurs in 1D fluid dynamics where it is used to describe the formation + propagation of shock and rarefaction waves

Let us consider the equation

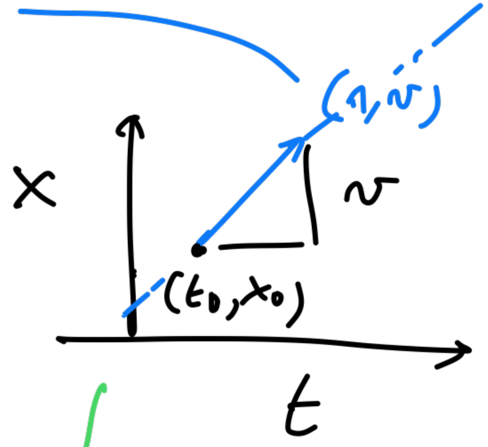
$$\boxed{c_t + v\, c_x = 0}$$



$c = c(x,t)$

characteristic

$\nabla c \cdot (1, v) = 0$



$\Rightarrow$ gradient of c $\perp$ $(1, v)$

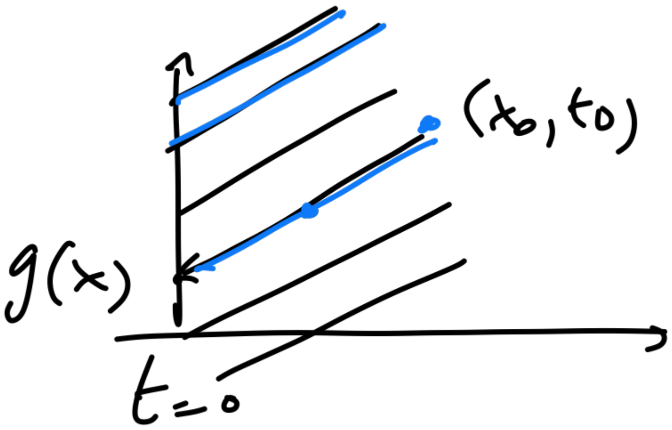
$\Rightarrow$ c constant along $(1, v)$

along the direction $(1, v)$ we have

$x = x_0 + v(t - t_0)$

$c(x_0 + v(t - t_0), t)$

$$w(t) = c(x_0 + v(t - t_0), t)$$



$$\dot{w}(t) = c_x \cdot v + c_t = 0$$

$$\Rightarrow w(t) = \text{cst}$$

Usually we are given the value at $t = 0$

$$c(x, t=0) = g(x) \quad *$$

for any (x_0, t_0) if we are given initial conditions of the form $(*)$

$$x = x_0 + v(t - t_0)$$

$$\downarrow$$

intersect the $\{t = 0\}$ axis at

$$x = x_0 - v t_0$$

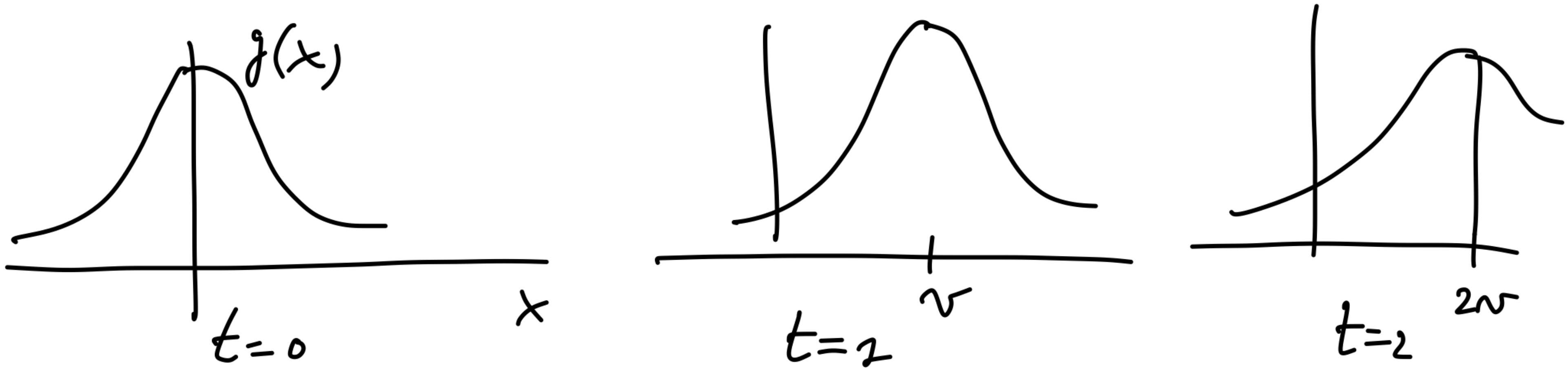
To find our solution (which is constant along the characteristic) we simply seek the ICs at the intersection of the characteristic with the $\{t=0\}$ axis

$$c(x_0, t_0) = g(x_0 - v t_0)$$

Since we did not make any assumption on x_0, t_0 we recover

$$c(x,t) = g(x - vt)$$

→ our Solution is thus a travelling wave which moves along +x direction



$$c_t + v c_x = f(x,t)$$ → concentration per unit time

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = -q(u(x_2,t)) + q(u(x_1,t)) + \int_{x_1}^{x_2} f(x,t)\,dx$$

$$c_t + v c_x = f \quad (**)$$

take c along the characteristic

let $w(t) = c(x_0 + v(t-t_0), t)$

substituting into $(**)$ we get

$$\dot{w}(t) = c_t + v c_x = f(x_0 + v(t-t_0), t)$$

$$w(t) = \underbrace{w(t=0)}_{\substack{\shortparallel \\ c(x_0 - v t_0, 0) = g(x_0 - v t_0)}} + \int_{t=0}^{t} f(x_0 + v(s-t_0), s)\, ds$$

$W(t)$ is the solution along the characteristic and so at (x_0, t_0) we get

$$\boxed{c(x_0,t_0) = g(x_0 - v t_0) + \int_0^t f(x_0 + v(s-t_0), s)\, ds}$$

<u>Proposition</u> let $g \in C^1(\mathbb{R})$ and $f, f_x \in C(\mathbb{R} \times (0,+\infty))$

the unique solution of the IC problem for (*)

$$\begin{cases} c_t + v c_x = f(x,t) & x \in \mathbb{R} \quad t>0 \\ c(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

is given by

$$c(x,t) = g(x - vt) + \int_0^t f(x - v(t-s), s)\, ds$$

→ 2 examples

→ quasi linear case

Example 1 decay in the concentration of Q

→ assume that Q decays with $r = -\gamma \cdot c(x,t)$

→ $$\begin{cases} c_t + v c_x = -\gamma c \\ c(x,0) = g(x) \end{cases}$$

take $u(x,t) = c(x,t) \cdot e^{\frac{\gamma}{v} x}$

$u_x = \left(c_x + c \frac{\gamma}{v} \right) e^{\frac{\gamma}{v} x}$ $\quad u_t = c_t e^{\frac{\gamma}{v} x}$

$$\begin{cases} u_t - v u_x = 0 \\ u(x,0) = C(x,0)\, e^{\frac{\gamma}{v}x} = g(x)\, e^{\frac{\gamma}{v}x} \end{cases}$$

$$u(x,t) = g(x-tv)\, e^{\frac{\gamma}{v}(x-tv)}$$

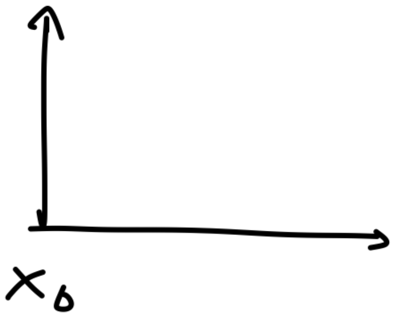
$$C(x,t) = g(x-tv)\, e^{-\gamma t}$$

Example 2 Assume a source active from time $t=0$

at $x=0$

whose concentration is kept constant

$$C(x=0,t)=\beta \quad \forall\, t>0$$



$$C(x,0)=0=g(x) \qquad x>0$$

$$\longrightarrow \quad C(x,t)=g(x-vt)$$

at time $t=0 \rightarrow C(x,t)=0 \Rightarrow g(x-vt)=0$

$$x>vt$$

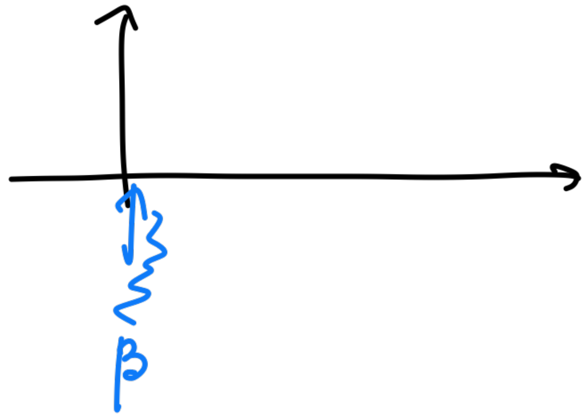
let us introduce

$$u(x,t)=C(x,t)\,e^{r/v\,x}$$

$$u_t+v\,u_x=0$$

$$u(x,t) = g(x - vt)$$

$$c(x,t) = g(\underbrace{x - vt})\, e^{-\frac{r}{v}x}$$



using $g(x) = 0$ for all $x > 0$

$$\Rightarrow u(x,t) = 0 \quad x > vt \;\rightarrow\; c(x,t) = 0 \quad x > vt$$

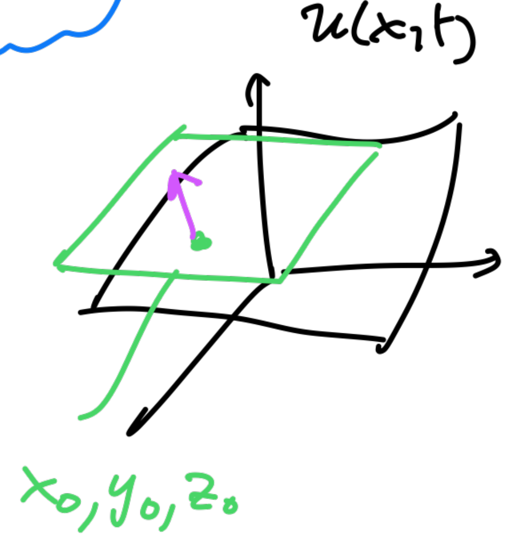
$$c(0,t) = g(-vt) = \beta \;\rightarrow\; g(x) = \beta \quad \forall x > 0$$

$$c(x,t) = \left\{ \begin{array}{ll} 0 & x > vt \\ \beta & x < vt \end{array} \right] e^{-r/v\, x}$$

Quasi-linear equations

$$\boxed{a(x,y,u)\,\overline{\frac{\partial u}{\partial x}} + b(x,y,u)\,\overline{\frac{\partial u}{\partial y}} = \overline{c}(x,y,u)} \quad (*)$$

→ tangent space to the graph of $u(x,t)$



$$u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0) - (z-z_0)$$

normal vector to the tangent plane is

$$[u_x(x_0,y_0),\ u_y(x_0,y_0),\ -1] = \vec{n}_0$$

the PDE shows that $\vec{n}_0$ is orthogonal to the

vector $\left[a(x,y,u),\ b(x,y,u)\ \ c(x,y,u) \right]$