First order equations → linear transport

→ quasilinear equation

→ non linear equations

$$F(x, y, u, u_x, u_y) = 0$$

+ assumption $F_{u_x}^2 + F_{u_y}^2 \neq 0$

in the quasilinear case we had

$$F(x, y, u, u_x, u_y) = a(x, y, u)\, u_x + b(x, y, u)\, u_y - c(x, y, u)$$

With $F_{u_x} = a(x,y,u)$ $\quad F_{u_y} = b(x,y,u)$

in this case we noticed that $\vec{v} = (a(x,y,u), b(x,y,u), c(x,y,u))$ was orthogonal to $(u_x, u_y, -1)$ which was itself the normal vector to the tangent space.

Following a similar idea, noting

$$\Big(F_{\overset{p}{u_x}}, F_{\overset{q}{u_y}}, \overbrace{-F_{u_x} u_x - F_{u_y} u_y}^{-F_p p - F_q q}\Big)$$ is orthogonal

to $(u_x, u_y, -1)$

From this using the idea of characteristics we get

$$\frac{dx}{dt} = F_p(x, y, u, p, q)$$

$$\frac{dy}{dt} = F_q(x, y, u, p, q)$$

$$\frac{dz}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = pF_p + qF_q$$

$$\frac{dp}{dt} = u_{xx} \frac{dx}{dt} + u_{xy} \frac{dy}{dt} = u_{xx} F_p + u_{xy} F_q$$

$$\frac{dq}{dt} = u_{yx} \frac{dx}{dt} + u_{yy} \frac{dy}{dt} = u_{yx} F_p + u_{yy} F_q$$

Since $F(x, y, u, u_x, u_y) = 0$

we also have $\tilde{F}_x(x, y, u, u_x, u_y) = 0$

$\tilde{F}_y(x, y, u, u_x, u_y) = 0$

derivative of the complete PDE with respect to x

$\neq F_x$

$\|$

derivative w.r.t to first argument

$$\tilde{F}_x = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial u_x}\frac{\partial u_x}{\partial x} + \frac{\partial F}{\partial u_y}\frac{\partial u_y}{\partial x} = 0$$

$$\tilde{F}_y = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial u_x}\frac{\partial u_x}{\partial y} + \frac{\partial F}{\partial u_y}\frac{\partial u_y}{\partial y} = 0$$

$$\tilde{F}_x = F_x + F_u p + F_p u_{xx} + F_q \cdot u_{yx} = 0$$

$$\tilde{F}_y = F_y + F_u q + F_p u_{xy} + F_q u_{yy} = 0$$

Substituting this in the characteristic equations, we get

$$\frac{dx}{dt} = F_p \qquad \frac{dy}{dt} = F_q \qquad \frac{dz}{dt} = pF_p + qF_q$$

$$\frac{dp}{dt} = -F_x - F_u p \qquad \frac{dq}{dt} = -F_y - F_u q$$

Note

$$\frac{d}{dt} F(x(t), y(t), z(t), p(t), q(t))$$

$$= F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_u \frac{dz}{dt} + F_p \frac{dp}{dt} + F_q \frac{dq}{dt}$$

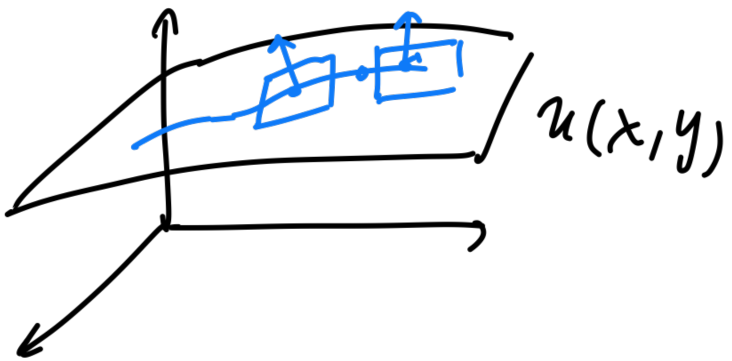
$$= F_x F_p + F_y F_q + F_u (pF_p + qF_q) + F_p(-F_x - F_u p) + F_q(-F_y - F_u q)$$

$p = \frac{\partial u}{\partial x}$ $\quad q = \frac{\partial u}{\partial y}$ $\qquad (u_x, u_y, -1)$

first two components of normal vector to tangent space

$\Rightarrow$ the characteristic vector (x, y, z, p, q) can be understood as providing not only the characteristic curve but also the orientation of the tangent plane

$\Rightarrow$ We call (x, y, z, p, q) a characteristic strip



As before, in order to derive the final expression of our characteristics we need an initial curve $\Gamma(s)$

However the IC's are most of the time given as the value of $u(x,y)$ along a curve

$$u(f(s), g(s)) = h(s)$$

In this case we also need conditions on p, q that is to say $\varphi(s) = u_x(f(s), g(s))$ $\psi(s) = u_y(f(s), g(s))$

In order to obtain a first equation involving $\varphi(s)$ and $\psi(s)$, we first note that our PDE must hold on $\Gamma(s)$

$$F(\overbrace{f(s)}, g(s), h(s), \varphi(s), \psi(s)) = 0$$

To obtain a second equation, note that we have

$$\underbrace{h'(s)}_{\text{change in } u \text{ along initial curve } \Gamma(s)} = \frac{du}{dx}\frac{dx}{ds} + \frac{du}{dy}\frac{dy}{ds} = \underbrace{\phi(s)}_{u_x(x(s),y(s))} f'(s) + \underbrace{\psi(s)}_{u_y(x(s),y(s))} . g'(s)$$

$\Rightarrow$ In Summary, to solve a First order PDE of the form $F(x, y, u, u_x, u_y) = 0$

Step 1 Solve for $\varphi(s), \psi(s)$ from the system

$$\begin{cases} F(f(s), g(s), h(s), \varphi(s), \psi(s)) = 0 \\ \varphi(s) f'(s) + \psi(s) g'(s) = h'(s) \end{cases}$$

Step 2 Solve the characteristic system

$$\frac{dx}{dt} = F_q \qquad \frac{dy}{dt} = F_q \qquad \frac{dz}{dt} = pF_q + qF_q$$

$$\frac{dp}{dt} = -F_x - pF_z \qquad \frac{dq}{dt} = -F_y - qF_z$$

With IC $\quad x(0) = f(s) \quad y(0) = g(s) \quad z(0) = h(s)$

$\quad p(0) = \varphi(s) \quad q(0) = \psi(s)$

Suppose we find the solution

$$x = X(t,s) \quad y = Y(t,s) \quad z = Z(t,s) \quad p = P(t,s) \quad q = Q(t,s)$$

Step 3 Solve for $x = X(t, s)$ $y = Y(t, s)$ for s, t in terms of x, y and substitute $s = S(x, y)$, $t = T(x, y)$ in $z = Z(s, t)$ to recover the final solution

$$z = z(x, y)$$

Just as in the linear and quasilinear cases, we can rely on the Implicit function theorem to derive a (local) existence result

Theorem

We consider the non linear first order PDE

$$F(x, y, u, u_x, u_y) = 0$$

Assume that

(i) F is twice continuously differentiable in a domain $D \subseteq \mathbb{R}^5$ and $F_p^2 + F_q^2 \neq 0$

(ii) f, g, h are twice continuously differentiable in a neighborhood of $s = 0$

(iii) (p_0, q_0) is a solution of the system

$$\begin{cases} F(x_0, y_0, z_0, p_0, q_0) \\ p_0 f'(0) + q_0 g'(0) = h'(0) \end{cases}$$

Where $(x_0, y_0, z_0) = (f(0), g(0), h(0))$

$$\begin{vmatrix} f'(0) & F_p(x_0, y_0, z_0, p_0, q_0) \\ g'(0) & F_q(x_0, y_0, z_0, p_0, q_0) \end{vmatrix} \neq 0$$

$\Rightarrow$ Then in a neighborhood of (x_0, y_0) there is a C^2 solution $z = u(x, y)$ of the Cauchy problem with initial data $x = f(s)$ $y = g(s)$ $z = h(s)$