Theorem [Global Max Principle] Evans PDEs

$$u \in C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$$

$$\begin{cases} u_t - D\Delta u = 0 & \text{on } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

and satisfies

$$|u(x,t)| \leq A e^{a|x|^2} \quad (x \in \mathbb{R}^n, \ 0 \leq t \leq T)$$

Moreover if we assume $T < \frac{1}{4aD}$ for some A, a, b

then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g(x)$$

Proof $T < \frac{1}{4aD} \Rightarrow \exists\, \varepsilon > 0$ s.t $4aD(T+\varepsilon) < 1$

$4aDT < 1$

fix $y \in \mathbb{R}^n$ $\mu > 0$

we consider the function $v(x,t)$ defined as

$$v(x,t) = u(x,t) - \frac{\mu}{D^{n/2}(T+\varepsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}} \quad \left(\begin{array}{l} x \in \mathbb{R}^n \\ t > 0 \end{array} \right)$$

Step 1 we will show that $\partial_t v - D\Delta v = 0$

$$\partial_t v = \partial_t u + \frac{\mu}{D^{n/2}} \left(-\frac{n}{2}\right) \frac{1}{(T+\varepsilon-t)^{n/2+1}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}}$$

$$+\frac{\mu}{D^{n/2+2}(T+\varepsilon-t)^{n/2}}|x-y|^2\frac{1}{4(T+\varepsilon-t)^2}e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}}$$

$$\partial_{x_i} v = \partial_{x_i} u - \frac{\mu}{D^{n/2+2}(T+\varepsilon-t)^{n/2}}\frac{2(x_i-y_i)}{4(T+\varepsilon-t)}e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}}$$

$$\partial_{x_ix_i} v = \partial_{x_ix_i} u - \frac{\mu}{D^{n/2+2}(T+\varepsilon-t)^{n/2}}\frac{2}{4(T+\varepsilon-t)}e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}}$$

$$-\frac{\mu}{D^{n/2+2}(T+\varepsilon-t)^{n/2}}\frac{4\,\overbrace{(x_i-y_i)}^{}{}^2}{16(T+\varepsilon-t)^2}e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}}$$

$$\Delta v = \Delta u - \frac{1}{2D^{n/2+2}}\frac{\mu\, n}{(T+\varepsilon-t)^{n/2+1}}e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}} - \frac{1}{4D^{n/2+2}}\frac{\mu}{(T+\varepsilon-t)^{n/2+2}}|x-y|^2 e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}}$$

Substitute $\partial_t v$ and Δv into Heat equation

shows $\quad \partial_t v - D\Delta v = 0$

Step 2 take $U = B(y, r)$ (ball centred at y of radius r)

From the Weak Maximum principle

$$\max_{\overline{Q}_T} v = \max_{\partial Q_T} v$$

In this case for v we have $\partial Q_T = \{t=0\} \cup \partial B(y,r)$

$$= \{t=0\} \cup \{|x-y| = r\}$$

$$v(x,0) = u(x,0) - \frac{\mu}{(T+\varepsilon)^{n/2} D^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon)D}}$$

$$\leq \underbrace{u(x,0)} = g(x)$$

On the $|x-y|=r$ part of the boundary we get

$$v(x,t) = \underbrace{u(x,t)} - \frac{\mu}{(T+\varepsilon-t)^{n/2} D^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)D}}$$

using $|u(x,t)| \leq A e^{b|x|^2}$

$$\leq A e^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2} D^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)D}}$$

use the fact that $\frac{1}{x^n} e^{1/x}$ is decreasing function of x : $\frac{d}{dx}\left(\frac{1}{x^n} e^{1/x}\right) = \frac{-n}{x^{n+1}} e^{\frac{1}{x}} + \frac{1}{x^n}\left(-\frac{1}{x^2}\right) e^{\frac{1}{x}}$

$\downarrow$

$$\leq A e^{b|x|^2} - \frac{\mu}{(T+\varepsilon)^{n/2} D^{n/2}} e^{\frac{r^2}{4(T+\varepsilon) D}} \quad (**)$$

Step 3 since $\frac{1}{4DT} > a$ and in particular

$$4D(T+\varepsilon)\, a < 1$$

$\exists\ \gamma$ s.t $\frac{1}{4D(T+\varepsilon)} = a+\gamma$

$|x|-|y| < r$

$|x-y| < r$

$|x| < r+|y|$

Substituting this into $(**)$

$$e^{b|x|^2} \leq e^{b(r+|y|)^2}$$

$$v(x,t) \le A e^{a(|y|+r)^2} - \mu\,(4(a+\gamma))^{3/2} e^{r^2(a+\gamma)}$$

$$\le \underbrace{e^{(a+\gamma)r^2}}\Big(-\mu\,(4(a+\gamma))^{3/2} + \underbrace{A e^{-\gamma r^2 + 2ar|y| + a|y|^2}}\Big)$$

For r large enough

$$\underbrace{v(x,t) \le \sup_{\mathbb{R}} g} \quad \text{on} \quad \overbrace{|x-y| = r}$$

taking the $\lim_{\mu \to 0}$ we get $u(x,t) \le \sup_{\mathbb{R}} g$ on $|x-y|=r$

Together $\overline{v(x,0)} \leq u(x,0) = g(x)$

$\underline{v(x,t)} \leq \underline{\sup g(x)}$ on $|x-y| = r$

$$\max_{\overline{Q_T}} v(x,t) \leq \sup g(x)$$

$\downarrow$ take $\mu \to 0$

$$\max_{\overline{Q_T} = B(y,r)} u(x,t) \leq \sup g(x)$$

$$\Rightarrow \sup_{\mathbb{R}^n \times [0,T]} u(x,t) \leq \sup g(x)$$