<u>Today</u> Laplace's equation

\+ Harmonic functions

→ mean value formula

→ Max principle

→ Invariant properties + Fundamental solution

→ Green Functions

Laplace's equation $\quad \Delta u = 0$

a function u is called harmonic on a domain $\Omega \subseteq \mathbb{R}^n$ if $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω

e.g.: equilibrium position of an elastic membrane
velocity of homogeneous fluid
steady state temperature of homogeneous + isotropic body.

Poisson equation $\quad \Delta u = f$

Motivation 1) Electrostatics $E = -\text{grad } V$

D = electric flux density $D = \varepsilon E$

Gauss' law $\nabla \cdot D = \rho_v$

$$\nabla(\varepsilon(-\nabla V)) = \rho_v$$

$$-\varepsilon \Delta V = \rho_v$$

Gravitation mass density $\rho(x)$

gravitational field $F(x)$

$\to$ F is expressed from the gravitational potential Φ as $F = -\nabla \Phi$

total mass $m = \int_V \rho(x)\,dx$

Gauss : $\oint_S F \cdot \vec{n}\,ds = -4\pi G m$

$-\oint_S \nabla\Phi \cdot \vec{n}\,ds = -4\pi G m$

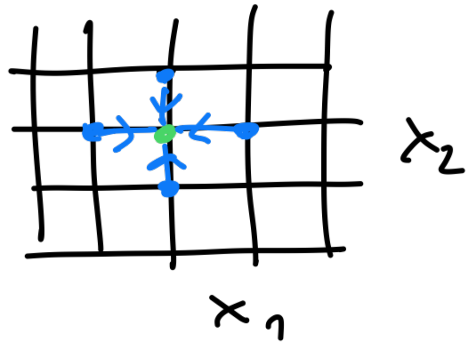
$= \int_V \nabla\cdot(\nabla\Phi)\,dV = 4\pi G m$ for every volume

$= 4\pi G \int \rho(x)\,dx$

$\Delta\Phi = 4\pi G \rho$

Towards Harmonic functions

$h\mathbb{Z}^2$



let us consider a particle moving on $h\mathbb{Z}^2$

at time steps τ

question: probability to observe particle at x after one step?

$$p(x, t+\tau) = \frac{1}{4}\left[p(x+he_1, t) + p(x-he_1, t) + p(x+he_2, t) + p(x-he_2, t) \right]$$

$$= \frac{1}{4} \sum_{|x-y|=h} p(y,t)$$

$u(x)$

$$= M_h\, p(x,t)$$

Mean value operator

$$\lim_{h\to 0} \frac{M_h u - u}{h^2} = \lim_{h\to 0} \frac{1}{h^2}(M_h - I)u$$

$$(M_h - I)u = \frac{1}{4}u(x+he_1) + \frac{1}{4}u(x-he_1) + \frac{1}{4}u(x+he_2) + u(x-he_2)\frac{1}{4}$$
$$- u(x)$$

using Taylor

$$u(x+he_i) = u(x) + h\, u_{x_i} + \frac{h^2}{2} u_{x_i x_i} + O(h^3) \quad \forall i$$

$$u(x-he_i) = u(x) - h\, u_{x_i} + \frac{h^2}{2} u_{x_i x_i} + O(h^3)$$

Substitute in (*)

$$(M_h - I)u = \underbrace{\frac{h^2}{2}\sum_{i=1}^{n}}\, u_{x_i x_i} + \underbrace{O(h^3)}$$

$$\left(\frac{M_h - I}{h^2}\right) = \Delta_h \longrightarrow \text{discrete laplace operator}$$

taking the limit of Δ_h when $h \to 0$

$$\lim_{h \to 0} \left(\frac{M_h - I}{h^2}\right) = \underbrace{\sum_{i=1}^{n} u_{x_i x_i}}$$

functions u satisfying $\Delta_h u = 0$ are called d-harmonic

Theorem (Mean value formulas)

Let u harmonic in $\Omega \subseteq \mathbb{R}^n$ for any ball $B_R(x) \subset\subset \Omega$

the following hold

(i) $$u(x) = \frac{n}{\omega_n R^n} \int_{B_R(x)} u(y)\, dy$$

(ii) $$u(x) = \frac{1}{\underline{\omega_n R^{n-1}}} \int_{\partial B_R(x)} u(\sigma)\, d\sigma$$

ω_n = surface of a radius 1 ball in $\mathbb{R}^n$

$$\omega_n = \frac{n \pi^{n/2}}{\Gamma\left(\frac{1}{2}n + 1\right)} \quad \text{where} \quad \Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t}\, dt$$

Euler-Gamma function

Proof For

$$g(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(\sigma)\, d\sigma$$

Surface element in $\mathbb{R}^n$ proportional to r^{n-1}

take the change of variable $\sigma \leftarrow \overbrace{x + r\sigma'}$

$$g(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(0,1)} u(x + r\sigma')\, d\sigma' \cdot r^{n-1}$$

3D r^2

2D $\to r$

$\sigma' = C$

$\sigma = r^{n-1} C$

$\sigma = \sigma' r^{n-1}$

$$= \frac{1}{\omega_n} \int_{\partial B(0,1)} u(x + r\sigma')\, d\sigma'$$

$$g'(r) = \frac{1}{\omega_n} \int_{\partial B(0,1)} \nabla_\xi u(\overbrace{x + r\sigma'}^{\xi}) \cdot \sigma'\, d\sigma'$$

$\sigma' = \frac{\xi - x}{r}$

Divergence theorem : $\displaystyle\int_V \Delta F \, dx = \oint_S F \cdot \vec{n} \, dS$

$$g'(r) = \frac{r}{\omega_n} \int_{\partial B(0,1)} \nabla_{\sigma'} u(x + r\sigma') \cdot \sigma' \, d\sigma'$$

$$= \frac{r}{\omega_n} \int_{B(0,1)} \underbrace{\Delta_{\sigma'} u(x + r\sigma')}_{} \, d\sigma' = 0$$

u is harmonic

$\Rightarrow g'(r) = 0$

$\Rightarrow g(r) = \text{cste}$

$$g(r) = \lim_{r \to 0^+} \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y) \, dy$$

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, dy$$

For (ii) using (i) we get

$$\int_0^R \omega_n r^{n-1} u(x)\, dr = \int_0^R \int_{\partial B_r(x)} u(y)\, dy$$

$$\omega_n \frac{R^n}{n} u(x) = \int_{B_R(x)} u(y)\, dy$$

$$u(x) = \frac{n}{\omega_n R^n} \int_{B_R(x)} u(y)\, dy$$