First Order Equations

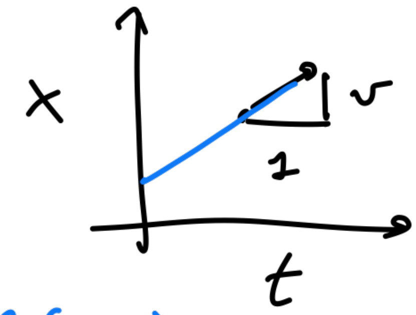
I Linear transport

Recap

$$u_t + v u_x = f$$

$\rightarrow$ $\nabla = (\partial_t, \partial_x)$

$$\nabla u \cdot (1, v)$$



$u(x,0) = g(x)$

$$x(t) = \bar{x} + v(t - \bar{t})$$

$$x = x_0 + v t$$

From this relation for any $\bar{x}, \bar{t}$

we get
$$\bar{x} = x_0 + v\bar{t}$$
$$x_0 = \bar{x} - v\bar{t}$$

Which we can use to recover the solution along the characteristic $u(\bar{x}, \bar{t}) = g(\bar{x} - v\bar{t})$

In case of a source term $u_t + v u_x = f$

we can study $u(x_0 + vt, t)$

from which we have $\frac{dw}{dt} = u_x \cdot v + u_t = f(x(t), t)$

$$\Rightarrow \quad w(t) - w(0) = \int_0^t f(x_0 + vs, s)\, ds$$

$$u(x_0 + vt, t) = \overbrace{u(x_0, 0)}^{g(x_0)} + \int_0^t f(x_0 + vs, s)\, ds$$

to get the solution u at any specific $(\bar{x}, \bar{t})$ we recover the intersection of the characteristic with the IC

$$x_0 = \underbrace{\bar{x} - v\bar{t}}$$

$$u(\bar{x}, \bar{t}) = g(\bar{x} - v\bar{t}) + \int_0^{\bar{t}} f(\bar{x} - v\bar{t} + vs, s)\, ds$$

II Extension to quasilinear equations

$$\boxed{a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)} \quad (*)$$

→ general idea of characteristics still applies

First note that tangent plane to graph $u(x,y)$ of u is given by

$$u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

this indicates that the vector $\vec{n}_0 = (u_x(x_0,y_0), u_y(x_0,y_0), -1)$ is normal to the tangent space

On the other hand from the PDE $(*)$ we see that this same vector is orthogonal to $(a(x,y,u), b(x,y,u), c(x,y,u))$

$\Rightarrow$ From this we can conclude that $(a(x,y,u), b(x,y,u), c(x,y,u))$ is tangent to the graph $u(x,y)$

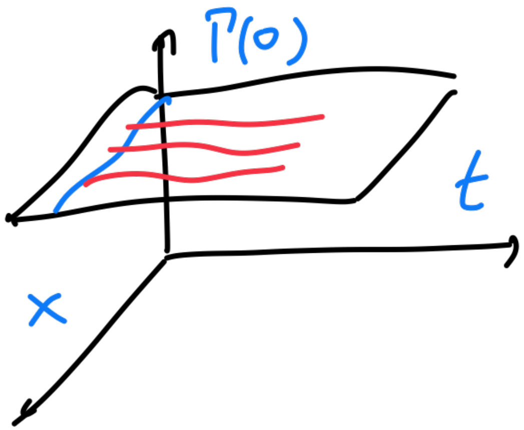
We can then try to reconstruct $u(x,y)$ by looking for the curves that are tangent to the vector field $(a(x,y,u), b(x,y,u), c(x,y,u))$

Doing this this leads to the system of ODEs

$$\frac{dx}{dt} = a(x,y,u) \qquad \frac{dy}{dt} = b(x,y,u) \qquad \frac{dz}{dt} = c(x,y,u)$$

which can be solved in combination with the initial conditions $u(x,0) = \phi(x)$

$\rightarrow$ introduce $\Gamma(s) = (s, 0)$ $u = \phi(s)$ on $\Gamma(s)$



$$\frac{dx}{dt} = a(x, y, u) \qquad x(0) = s$$

$$\frac{dy}{dt} = b(x, y, u) \qquad y(0) = 0$$

$$\frac{dz}{dt} = c(x, y, u) \qquad z(0) = \phi(s)$$

→ We saw that u could be uniquely recovered provided that we could invert the mapping from $(t, s) \mapsto (x, y)$ in a way such that $s(x,y)$, $t(x,y)$ were well defined.

III We saw that in certain settings, the method of characteristics was unable to provide a satisfactory solution.

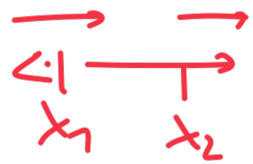
→ III a) for initial conditions exhibiting a discontinuity, we saw that solution obtained by the characteristics might not be defined on the whole space $\{x \geq 0\} \times \{t \geq 0\}$ in this case we saw one solution which consisted in introducing <u>fan like characteristics</u>

→ such characteristics led to a solution known as rarefaction wave / simple wave.

→ III b) Another setting in which the method of characteristics fails is when the characteristics overlap

→ in such a setting it makes sense to consider the introduction of a discontinuity curve $s(t)$

From which we have to split our conservation law as



$$\frac{d}{dt}\left\{\int_{x_1}^{s(t)} u(y,t)\,dy + \int_{s(t)}^{x_2} u(y,t)\,dy\right\} = -q(u(x_2,t)) + q(u(x_1,t)) \qquad (*)$$

$$\frac{d}{dt}\int_{x_1}^{s(t)} u(y,t)\,dy = \int_{x_1}^{s(t)} u_t(y,t)\,dy + \lim_{y\to s-} u(\overset{y}{\cancel{s(t)}},t)\cdot \dot{s}(t)$$

$$\frac{d}{dt}\int_{s(t)}^{x_2} u(y,t)\,dy = \int_{s(t)}^{x_2} u_t(y,t)\,dy - \lim_{y\to s+} u(\overset{y}{\cancel{s(t)}},t)\ \dot{s}(t)$$

Substituting this in (*)

$$\int_{x_1}^{s(t)} u_t(y,t)\,dy + \int_{s(t)}^{x_2} u_t(y,t)\,dy + u^-(y,t)\,\dot{s}(t) - u^+(y,t)\,\dot{s}(t)$$

$$= -q(u(x_2,t)) + q(u(x_1,t))$$

taking $\lim_{x_1 \to s^-}$, $\lim_{x_2 \to s^+}$ the integrals both vanish

and we are left with

$$(u^- - u^+)\,\dot{s}(t) = -q(u^+) + q(u^-)$$

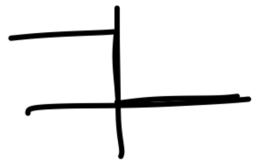
$$\Rightarrow \quad \dot{s}(t) = \frac{q(u^-) - q(u^+)}{u^- - u^+}$$

let us consider the IVP

$$u_t + q(u)_x = 0 \quad \rightarrow \quad q(u)_x = 2u\,u_x$$

$s(t)$ $\quad q(u) = u^2$

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0 \qquad u(x,0) = \begin{cases} 4 & x<0 \\ 3 & x>0 \end{cases}$$



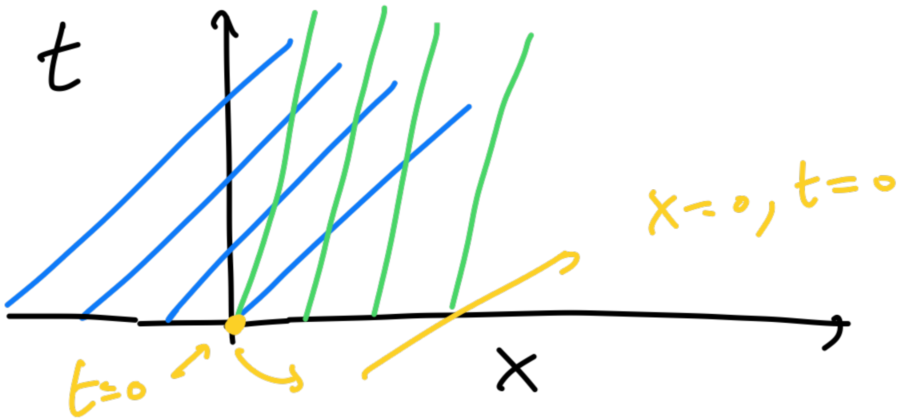
$$\frac{dt}{d\xi} = 1 \qquad \frac{dx}{d\xi} = 2u \qquad \frac{dz}{d\xi} = 0 \qquad \Gamma(s) = (s,0)$$

$$\phi(s) = \begin{cases} 4 & s<0 \\ 3 & s>0 \end{cases}$$

$$t(0) = 0 \qquad x(0) = s \qquad z(0) = \phi(s)$$

$$z = \phi(s)$$

$$\rightarrow t = \xi \qquad x = 2\phi(s)\xi + s$$



$$x(t) = 2\phi(s)\,t + s$$

$$= \begin{cases} 8t + x_0 & x_0 < 0 \\ 6t + x_0 & x_0 > 0 \end{cases}$$

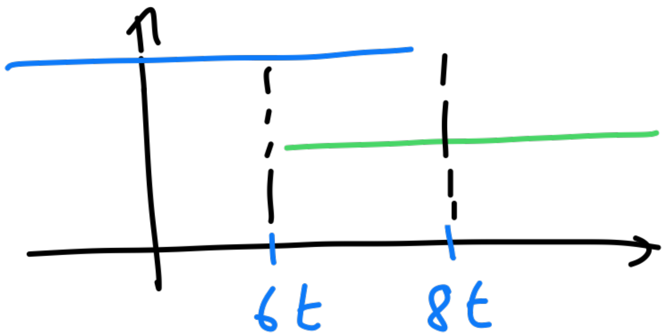
To recover the solution, inverting x and y we get

$$s(x,y) = x - 2\phi(s)t = \begin{cases} x - 8t & x - 8t < 0 \\ x - 6t & x - 6t > 0 \end{cases}$$

$$\xi = t$$

$$u(x,t) = \phi(s(x,t)) = \begin{cases} 4 & s(x,t) < 0 \\ 3 & s(x,t) > 0 \end{cases}$$

$$= \begin{cases} 4 & x < 8t \\ 3 & x > 6t \end{cases}$$



in this case, introducing the discontinuity curve $s(t)$ and getting the value of $\dot{s}(t)$ from the Rankine Hugoniot,

$$u^+ = 3$$
$$u^- = 4$$

$$q(u) = u^2 \Rightarrow q(u^+) = 9 \quad q(u^-) = 16$$

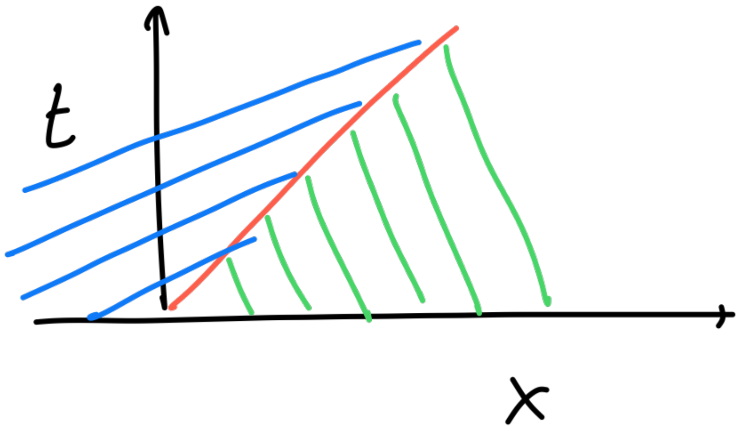
$$\dot{s}(t) = \frac{16-9}{1} = 7$$

$$s(t) = 7t + C$$

$s(0) = 0 \rightarrow$ first point (x, t) at which characteristics overlap

$$s(t) = 7t$$

together, we get


t
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