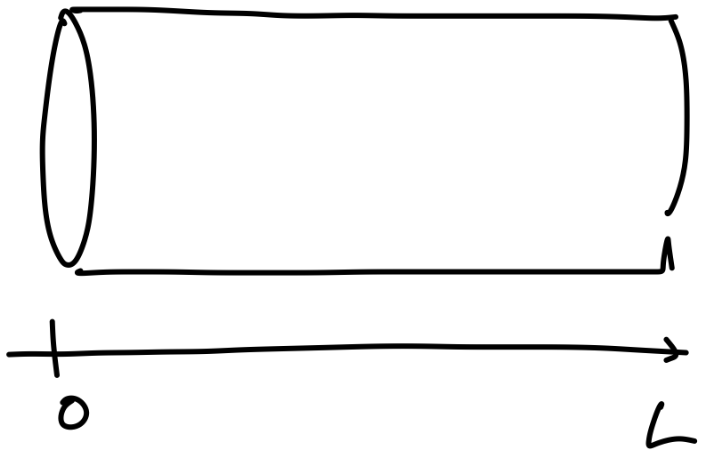
Heat equation

$$\frac{\partial u}{\partial t} = \frac{k_0}{c\rho} \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$



+ initial conditions $u(x,0) = g(x)$

+ boundary conditions $\begin{cases} u(0,t) = h_1(t) \\ u(L,t) = h_2(t) \end{cases}$ for $t \in (0,T]$

1) Dirichlet BCs : $u(0,t) = h_1(t)$, $u(L,t) = h_2(t)$

2) Neumann BCs : specify heat flux

$$x = 0 \; : \; -K_0 u_x(0,t)$$
$$x = L \; : \; -K_0 u_x(L,t)$$

3) Robin BCs (surroundings are kept at temperature U and we assume that inward heat flux depends linearly on $U - u$ (Newton's law of cooling)

$$K_0 u_x = \gamma (U - u) \qquad (\gamma > 0)$$

$$\alpha = \gamma / K_0 \qquad h = \frac{\gamma U}{K_0} \quad \rightarrow \quad u_x + \alpha u = h$$

4) Any combination of the first 3

Given $f = f(x,t)$ (source term) and $g = g(x)$ initial Cauchy data, find a function $u(x,t)$ that satisfies

$$\begin{cases} u_t - D u_{xx} = f & 0 < x < L \quad 0 < t < T \\ u(x,0) = g(x) & 0 \leq x \leq L \\ + \text{ boundary conditions} & 0 \leq t < T \end{cases}$$

Cauchy - Dirichlet Problem for the heat equation

BCs can be

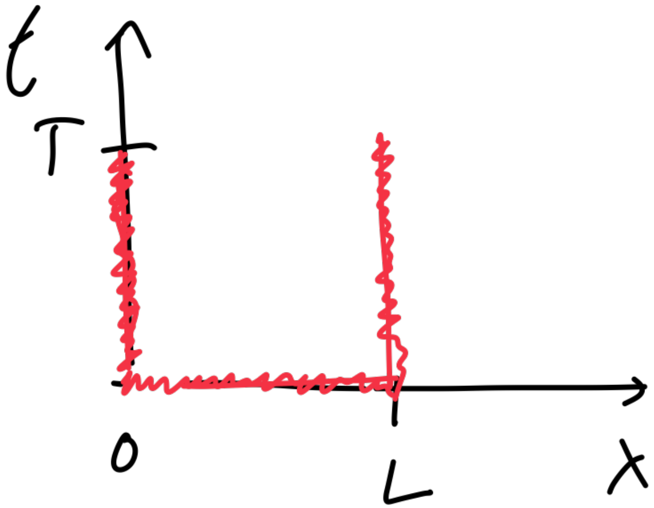
<u>Dirichlet</u> $u(0,t) = h_1(t) \quad u(L,t) = h_2(t)$

<u>Neumann</u> $-u_x(0,t) = h_1(t) \quad u_x(L,t) = h_2(t)$

<u>Robin or radiation</u> $-u_x(0,t) + \alpha u(0,t) = h_1(t)$

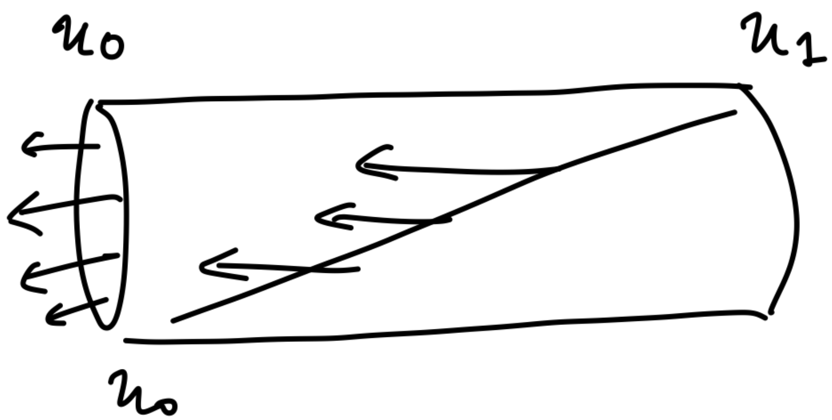
$u_x(L,t) + \alpha u(L,t) = h_2(t)$

+ <u>any combination</u> (mixed conditions)



part of the boundary specified = <u>parabolic boundary</u>

<u>Today</u> : Method of separation of variables



$u(x,0) = u_0$

$u(L,t) = u_1 \quad t > 0$

$u(0,t) = u_0$

1) steady state solution

2) Complete solution

$$\begin{cases} u_t - D u_{xx} = 0 \\ u(x,0) = u_0 \\ u(0,t) = u_0 \\ u(L,t) = u_1 \end{cases}$$

- Bar is of length L

→ replace x with dimensionless $\frac{x}{L} = y$

since $x \in [0, L] \rightarrow y \in [0, 1]$

- to get a characteristic time, note the Diffusion constant D has units $\frac{[\text{length}]^2}{[\text{time}]}$

$\Rightarrow$ we can get a characteristic time by defining $\tau = \frac{L^2}{D}$ $\quad \tau\,[\text{time}]$

From this, we can define our second dimensionless variable as

$$\delta = \frac{t}{\tau}$$

- To get dimensionless formulation we define

$$Z(y,s) = \frac{u(Ly, \tau s) - u_0}{u_1 - u_0}$$

Cauchy / Dirichlet

$u(x,0) = u_0$

$u(0,t) = u_0$

$u(L,t) = u_1$

$\longrightarrow$

$$\begin{cases} Z(y,0) = 0 \\ Z(0,s) = 0 \\ Z(1,s) = 1 \end{cases}$$

$$u(x,t) = (u_1 - u_0)\, z\left(\frac{x}{L}, \frac{t}{\tau}\right) + u_0$$

$$\frac{\partial u}{\partial t} = (u_1 - u_0) \frac{\partial z}{\partial s} \cdot \frac{\partial s}{\partial t} = (u_1 - u_0) \frac{\partial z}{\partial s} \cdot \frac{1}{\tau}$$

$$\frac{\partial u}{\partial x} = (u_1 - u_0) \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x} = (u_1 - u_0) \frac{\partial z}{\partial y} \cdot \frac{1}{L}$$

$$\frac{\partial^2 u}{\partial x^2} = (u_1 - u_0) \frac{\partial^2 z}{\partial y^2} \cdot \left(\frac{1}{L}\right)^2$$

Dimensionless formulation

$$
\begin{cases}
\cancel{(u_1 - u_0)} \dfrac{\partial z}{\partial s} \dfrac{1}{\tau} - D \cancel{(u_1 - u_0)} \dfrac{\partial^2 z}{\partial y^2} \left(\dfrac{1}{L}\right)^2 = 0 \\
z(y, 0) = 0 \\
z(0, s) = 0 \\
z(1, s) = 1
\end{cases}
$$

recall that $\tau = \dfrac{L^2}{D}$

$$
\left(
\begin{cases}
\dfrac{\partial z}{\partial s} - \dfrac{\partial^2 z}{\partial y^2} = 0 \\
z(y, 0) = 0 \\
z(0, s) = 0 \\
z(1, s) = 1
\end{cases}
\right)
$$

Dimensionless problem

let us start with the steady state solution z^{st}. this solution satisfies

$$\begin{cases} \frac{\partial^2 z}{\partial y^2} = z_{yy} = 0 \\ z(0) = 0 \\ z(1) = 1 \end{cases}$$

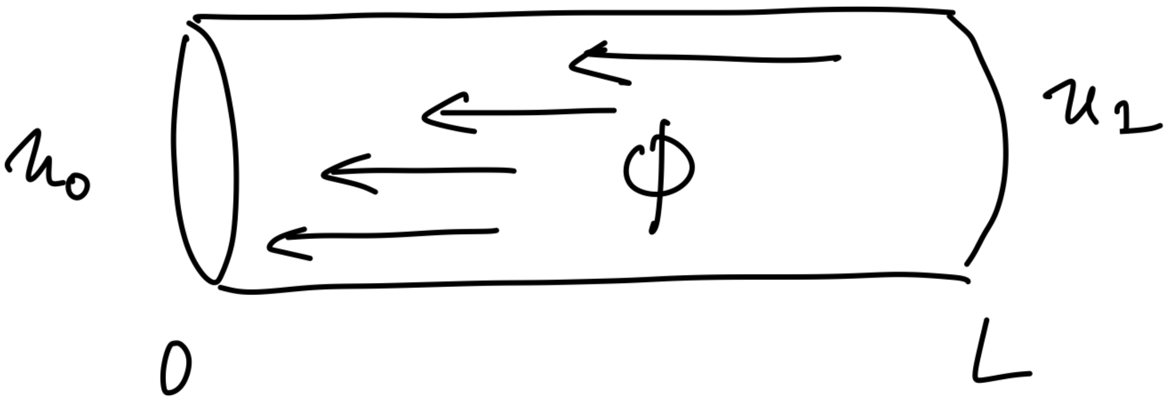
$$\Rightarrow \frac{d^2 z}{dy^2} = 0 \Rightarrow Ay + B = z(y)$$

using BC's $\Rightarrow \begin{cases} z(0) = 0 \Rightarrow B = 0 \\ z(1) = 1 \Rightarrow A = 1 \end{cases}$

together we get $z^{st}(y) = y$

using the mapping between z and u, we get

$$u^{st}(x) = (u_1 - u_0)\frac{x}{L} + u_0$$



$$\phi = -K_0 \frac{\partial u}{\partial x}$$

$$= -K_0 \frac{(u_1 - u_0)}{L}$$

Transient regime

1) define $U(y,s) = z^{st}(y,s) - z(y,s)$ ← Complete solution of Cauchy Dirichlet Problem

$$= y - z(y,s)$$

$$z^{st}_{yy} = 0 = z^{st}_s$$

$U(y,s) \to 0$ as $s \to \infty$

Since both $z(y,S)$ & $z^{st}(y)$ satisfy the heat equation, so does $U(y,S)$

$$U_s - U_{yy} = 0$$

on $z(y,S) \to$

$$z(y,0) = 0 \qquad\qquad U(y,0) = y$$

$$z(0,S) = 0 \quad \longrightarrow \quad U(0,S) = 0$$

$$z(1,S) = 1 \qquad\qquad U(1,S) = 0$$

$U(y,s)$ in this case is known as a transient regime.

$$\begin{cases} U_s - U_{yy} = 0 \\ U(y,0) = y \\ U(0,s) = 0 \\ U(1,s) = 0 \end{cases} \quad (*)$$

→ To find an explicit formula for $U(y,s)$ we will use the method of separation of variables

general idea: write $U(y,s)$ as

$$U(y,s) = v(y)\, w(s)$$

let us substitute this in $(*)$

$$\begin{cases} v(y)\,W'(s) - v''(y)\,W(s) = 0 \\ \begin{cases} v(0) = 0 \\ v(1) = 0 \end{cases} \end{cases}$$

+ initial condition

$$\frac{v''(y)}{v(y)} = \frac{W'(s)}{W(s)} \quad \text{for every } y \text{ and } s$$

$\Rightarrow$ only possibility : $\dfrac{v''(y)}{v(y)} = \lambda = \dfrac{W'(s)}{W(s)}$

let us start with the equation $\dfrac{v''(y)}{v(y)} = \lambda$ eigenvalue problem

eigenvalue $= \lambda$

• either $\lambda = 0$ in this case

we get $v''(y) = 0 \Rightarrow v(y) = Ay + B$

using the boundary conditions

$v(0) = 0$ $\longrightarrow$ $A = B = 0$

$v(1) = 0$

μ^2

$\|$

• or $\lambda > 0$ in this case

we get $v''(y) = \lambda v(y)$

$$v(y) = A e^{\mu y} + B e^{-\mu y}$$

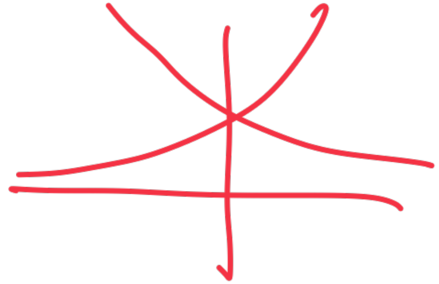
+ Boundary conditions : $v(0) = 0 \Rightarrow A + B = 0$

$v(1) = 0 \Rightarrow A e^{\mu} + B e^{-\mu} = 0$

$) \quad A = -B$

$$\left\{ Ae^{\mu} - Ae^{-\mu} = 0 \right.$$

$$A(e^{\mu} - e^{-\mu}) = 0$$



$$\rightarrow A = B = 0$$

- $\lambda = -\mu^2 < 0$

$$v''(y) = -\mu^2 v(y)$$

$$\rightarrow v(y) = Ae^{i\mu y} + Be^{-i\mu y}$$

Boundary conditions $v(0) = 0$ $\quad A + B = 0$

$v(1) = 0$

$$Ae^{i\mu} - Ae^{-i\mu} = 0$$

$$2A \sin \mu = 0 \implies \sin \mu = 0$$

$$\mu = k\pi \qquad k = 1, 2, \ldots$$

$$v_k(y) = A_k \sin(k\pi y)$$

eigen functions

We are now left with the equation

$$\frac{W'(s)}{W(s)} = -\mu^2 \rightarrow W(s) = A e^{-\mu^2 s}$$

$$W(s) = A e^{-(k\pi)^2 s}$$

Combining $v(y)$ and $W(s)$, we get $U(y, s)$ as

$$\underbrace{U_k(y,s)} = A_k e^{-(k\pi)^2 s} \sin(k\pi y)$$

the general solution of (*)

$$U(y,s) = \sum_{k=1}^{\infty} A_k e^{-(k\pi)^2 s} \sin(k\pi y)$$

To satisfy the IC, we take

$$U(y,0) = \sum_{k=1}^{\infty} A_k \sin(k\pi y) = y$$

) Can we select A_k's so that the equality hold?

We now face a number of questions

1) Can we find values for the A_k such that $U(y,0) = y$?

2) Any finite combination of eigen solutions will still be a solution

How about an ∞ sum?

$$(\partial_s - \partial_{yy})\, U(y,s) \stackrel{?}{=} \sum_{k=1}^{\infty} (\partial_s - \partial_{yy})\, U_k$$

3) In what sense does $U(y,s)$ satisfy the initial conditions?

Question 1 : answer related to the notion of Fourier series.

When dealing with the heat, we discovered that the initial conditions could be satisfied only if $f(x)$ could be written as a linear combination of eigenfunctions (sines)

Let u be a $2T$-periodic function and assume

$$u(x) = U + \sum_{k=1}^{\infty} \left\{ a_k \cos \frac{2\pi k}{T} x + b_k \sin \frac{2\pi k}{T} x \right\}$$

Question? What is the expression of a_k, b_k, U?

Key idea: use orthogonality of the trigonometric functions

$\omega = \frac{\pi}{T}$

$$\int_{-T}^{T} \cos k\omega x \cos m\omega x \, dx = 0 \quad \text{if } k \neq m$$

$$\int_{-T}^{T} \sin k\omega x \sin m\omega x \, dx = 0 \quad \text{if } k \neq m$$

$$\int_{-T}^{T} \cos k\omega x \sin m\omega x \, dx = 0 \quad \text{for all } k, m$$

$$\int_{-T}^{T} \cos^2 k\omega x \, dx = T$$

$$\int_{-T}^{T} \sin^2 k\omega x \, dx = T$$

let $u(x) = U + \sum_{k=1}^{\infty} \left\{ a_k \cos \frac{2\pi k}{T} x + b_k \sin \frac{2\pi k}{T} x \right\}$

To get the expression of any coefficient a_k (or b_k) we can use the orthogonality of the trigonometric functions by multiplying both sides with $\cos \frac{2\pi k}{T}$ (or $\sin \frac{2\pi k}{T}$) and then integrate on $[-T, T]$.

take $l > 0$

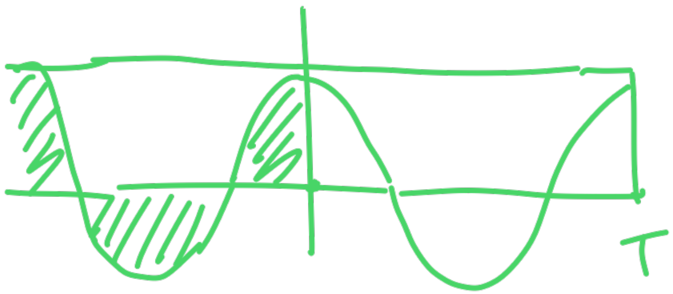
e.g $\int_{-T}^{T} u(x) \cos \frac{2\pi l}{2T} x \, dx$

$$= \int_{-T}^{T} U \cos \frac{2\pi l}{2T} x \, dx + \int_{-T}^{T} \sum_{k=1}^{\infty} a_k \cos \frac{2\pi k}{2T} x \cos \frac{2\pi l}{2T} x \, dx$$

$$+ \int_{-T}^{T} \sum_{k=1}^{\infty} b_k \sin \frac{2\pi k}{2T} x \cos \frac{2\pi l}{2T} x \, dx = 0$$

$$\int_{-T}^{T} U \cos \frac{2\pi l}{2T} x \, dx = U \frac{T}{2\pi l} \left| \sin \frac{2\pi l}{2T} x \right|_{-T}^{T}$$

$$= 0$$



$$a_l \int_{-T}^{T} \cos^2 \frac{2\pi k}{2T} x \, dx$$

$$= a_l T$$

$$\int_{-T}^{T} u(x) \cos \frac{2\pi l}{2T} x \, dx = a_l T$$

Provided that $u(x)$ has an expansion as an combination of trigonometric function then

$$a_l = \frac{1}{T} \int_{-T}^{T} u(x) \cos \frac{2\pi l}{2T} x \, dx$$

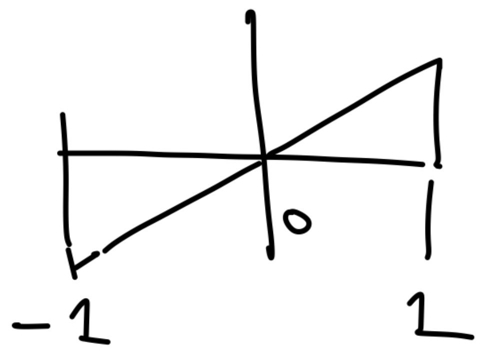
$$b_l = \frac{1}{T} \int_{-T}^{T} u(x) \sin \frac{2\pi l}{2T} x \, dx$$

$$U = \frac{1}{T} \int_{-T}^{T} u(x)$$

Going back to the Heat equation recall that the ICs

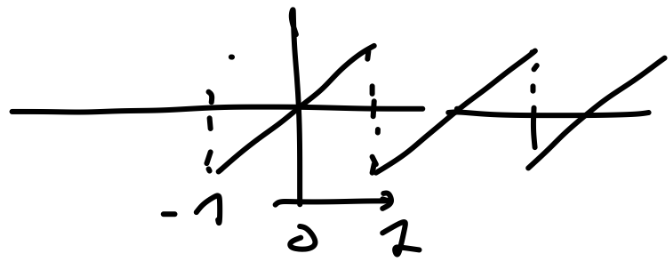
$$U(y,0) = \sum_{k=1}^{\infty} A_k \sin \pi k y = y$$

→ from orthogonality of trigonometric functions we now know that to satisfy these ICs it suffices to take



$$A_k = \int_{-1}^{1} y \sin \pi k y \, dy$$

$$= \int_{0}^{1} y \sin \pi k y \, dy$$



→ Complete final solution as

$$U(y,s) = \sum_{k=1}^{\infty} A_k \, e^{-k^2\pi^2 s} \sin k\pi y$$

Step 1 : Making sure that IC's are satisfied

to enforce our I.C, we expanded y into a Fourier series and then we did match the coefficients of expansion of U and the expansion of y

For y we had

$$y = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k\pi} \sin k\pi y$$

$x \in [0, L]$

$y \in [0, 1]$

$$\lim_{s \to 0} \int_0^1 [U(y,s) - y]^2 \, dy = 0$$

by Parseval : $\int_0^1 [U(y,s) - y]^2 \, dy$

using $U(y,s) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k\pi} e^{-k^2\pi^2 s} \sin k\pi y$

$$\int_0^1 [U(y,s) - y]^2 \, dy = \sum_{k=1}^{\infty} \frac{4}{\pi^2} \left(\frac{e^{-k^2\pi^2 s} - 1}{k^2} \right)^2$$

$$\leq \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

since the infinite series converges, we can take the limit past the sum

Parseval

$$\lim_{s \to 0} \int_0^1 [U(y,s) - y]^2 \, dy = \lim_{s \to 0} \sum_{k=1}^{\infty} \frac{4}{\pi^2} \left(\frac{e^{-k^2\pi^2 s} - 1}{k^2} \right)^2$$

$$= \sum_{k=1}^{\infty} \lim_{s \to 0} \frac{4}{\pi^2} \left(\frac{e^{-k^2\pi^2 s} - 1}{k^2} \right)^2 = 0$$

Initial conditions satisfied in L^2 sense,

$$\lim_{s \to 0} \int_0^1 |e(y,s)|^2 \, dy \to 0 \qquad e(y,s) = U(y,s) - y$$

Step 2 Derivatives $\frac{\partial U}{\partial s}$, $\frac{\partial^2 U}{\partial y^2}$ must be well defined

$$\sum_{k=1}^{\infty} \underbrace{(-1)^{k+1} \frac{2}{k\pi} e^{-k^2\pi^2 s} \sin k\pi y}_{U_k}$$

$$\left|\frac{\partial U_k}{\partial s}\right| \quad \left|\frac{\partial^2 U_k}{\partial y^2}\right| < 1 \, \frac{2}{k\pi} k^2 \pi \leq 2k\pi \, e^{-k^2\pi^2 s_0}$$

the series $\sum_{k=1}^{\infty} 2k\pi e^{-k^2\pi^2 s_0}$ converges for $s \geq s_0 > 0$

$$\left(\text{to see this, check the ratio } \lim_{k \to \infty} \frac{(k+1) e^{-(k+1)^2\pi^2 s_0}}{k \, e^{-k^2\pi^2 s_0}} \right)$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{\partial U_k}{\partial s} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\partial^2 U_k}{\partial y^2} \quad \text{both converge}$$

$$\rightarrow \quad \frac{\partial U}{\partial s}, \quad \frac{\partial^2 U}{\partial y^2} \quad \text{are well defined.}$$

Step 3 for BC's

since the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k\pi} e^{-k^2\pi^2 s} \sin k\pi y$

for all y, $s_0 > 0$

$$\lim_{(y,s)\to(0,s_0)} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k\pi} e^{-k^2\pi^2 s} \sin k\pi y$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k\pi} e^{-k^2\pi^2 s_0} \cdot 0$$