# MATH-UA 9263 - Partial Differential Equations Recitation 5: Poisson equation, Green function and Green representation formula

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**Question 1** *We want to find the solution to the following Poisson problem*

$$u_{xx} + u_{yy} = f(x, y) \tag{1}$$

*in the domain* $\left\{(x, y), x > 0, y > 0, x^2 + y^2 < 1\right\}$ *with boundary conditions*

$$u = 0 \quad \text{for } y = 0,\ 0 < x < 1 \tag{2}$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{for } x = 0,\ 0 < y < 1, \tag{3}$$

$$u = 0 \quad \text{for } x > 0,\ y > 0,\ x^2 + y^2 = 1 \tag{4}$$

*Where* $f(x, y)$ *is known and is assumed to be continuous. [Hint: first find the eigenfunction in* $\theta$ *then substitute the expression* $u(\theta, r) = v(\theta)w(r)$ *in the equation and solve the ODE for* $w(r)$*.]*

**Question 2** *Using Green representation formula together with the expression of the Green function on the sphere, derive Poisson's formula for the unique solution of the Dirichlet problem* $\Delta u = 0$ *in* $B_R(\mathbf{0})$ *with* $u = g$ *on* $\partial B_R(\mathbf{0})$.

**Question 3** *Using the method of images, solve*

$$\Delta G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

*in the first quadrant (*$x \geq 0$ *and* $y \geq 0$*) with* $G = 0$ *on the boundaries.*

**Question 4** *Use the method of image to obtain the Green function*

$$\Delta G(\boldsymbol{x}; \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{5}$$

1. *On the infinite strip* $0 < x < L$, $-\infty < y < \infty$ *if* $G = 0$ *at* $x = 0$ *and* $\frac{\partial G}{\partial x} = 0$ *at* $x = L$
2. *On the infinite strip (*$0 < x < L$, $-\infty < y < \infty$, $-\infty < z < \infty$*) if* $G = 0$ *at* $x = 0$ *and* $G = 0$ *at* $x = L$

3. *On the semi-infinite strip ($0 < x < L$ and $0 < y < \infty$) if $G = 0$ along the boundaries.*

**Question 5** *Prove the following properties of the Green function*

1. $G(\boldsymbol{x}, \boldsymbol{y}) > 0$ *(positivity)*
2. $G(\boldsymbol{x}, \boldsymbol{y}) = G(\boldsymbol{y}, \boldsymbol{x})$ *(symmetry)*
3. $G(\boldsymbol{x}, \boldsymbol{y}) \leq \Phi(\boldsymbol{x}, \boldsymbol{y})$

*[Hint (1): For any $x \in \Omega$, let $G(x, y)$ denote the Green function centered at $x$. Consider a ball $B_r(x)$ around $x$. Then use Harnack's inequality as well as the fact that harmonic functions acheive their extremas on the boundary (this can be proved in a similar fashion as Harnack's inequality, by using the mean value formula and a covering argument). (2) For any fixed $x \in \Omega$, define $u(y) = G(x, y)$ and $v(y) = G(y, x)$ then use Green's identity in $\Omega \setminus B_r(\boldsymbol{x})$ and take $r \to 0$]*

**Question 6** *Find the one dimensional Green's function for the interval $(0, \ell)$. The three properties defining it can be restated as follows*

(i) *It solves $G''(x) = 0$ for $x \neq x_0$ (harmonic)*

(ii) $G(0) = G(\ell) = 0$

(iii) *$G(x)$ is continuous at $x_0$ and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at $x_0$*

**Question 7** *Find the Green function for the tilted half-space $\{(x, y, z) | ax + by + cz > 0\}$*

**Question 8** *The Neumann function $N(x, y)$ for a domain $D$ is defined exactly like the Green's function except that the condition $G(\boldsymbol{x}, \boldsymbol{y}) = 0$ on $\partial\Omega$ is replaced by $\frac{\partial N}{\partial n} = c$ for $x \in \partial\Omega$ for a suitable constant $c$.*

(i) *Show that $c = 1/A$ where $A$ is the area of $\partial\Omega$ ($c = 0$ if $A = \infty$)*

(ii) *State and prove an analog of Green's representation formula*

$$u(\boldsymbol{x}) = -\int_{\Omega} f(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y} - \int_{\partial\Omega} g(\boldsymbol{\sigma}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{\sigma})}{\partial \boldsymbol{\nu}} d\boldsymbol{\sigma} \tag{6}$$

*(that we derived for the Dirichlet problem with boundary data $g$), that expresses the solution of the Neumann problem from the Neumann function.*