Partial Differential Equations, lecture 5

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The Cauchy problem

In this lecture, we will study the Cauchy problem

$$\begin{cases} u_t - Du_{xx} = 0, & \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x) & \text{in } \mathbb{R} \end{cases}$$

We start by proving existence of a solution. We consider the candidate solution given by

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) g(y)\, dy$$

where $\Phi(x-y)$ is the fundamental solution. Such a candidate solution is motivated by the fact that, as we previously saw, $\lim_{t\to 0^+} \Phi(x-y,t) = \delta(x-y)$ and

$$g(x) = \int_{\mathbb{R}} \delta(x-y) g(y)\, dy \tag{1}$$

From those we thus have

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \int_{\mathbb{R}} \Phi(x-y,t) g(y)\, dy = g(x)$$

Moreover, since $\Phi(x,t)$ represents the fundamental solution, hence satisfies

$$\int_{\mathbb{R}} (\partial_t \Phi(x-y,t) - D\Delta\Phi(x-y,t)) g(y)\, dy = 0. \tag{2}$$

we can expect our candidate solution u to verify

$$\partial_t u - D\Delta u = 0. \tag{3}$$

Provided that we can move the differential operator inside the integral, it therefore seems like u provides a valid solution.

The following theorem shows that our reasoning is perfectly valid when the Cauchy data $g(x)$ satisfies an exponential growth condition.

Theorem 1 (Existence of a solution for the Cauchy problem). *Assume that there exist positive numbers a and c such that*

$$|g(x)| \leq ce^{ax^2}, \quad \textit{for all } x \in \mathbb{R} \tag{4}$$

Let u be given by

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)g(y)\, dy = \frac{1}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4Dt}} g(y)\, dy \tag{5}$$

and $T < \frac{1}{4aD}$. Then, the following properties hold

(i) There are positive numbers C and A such that

$$|u(x,t)| \leq Ce^{Ax^2}, \quad \textit{for all } (x,t) \in \mathbb{R} \times (0,T] \tag{6}$$

(ii) $u \in C^\infty(\mathbb{R} \times (0,T])$ and in the strip $\mathbb{R} \times (0,T]$

$$u_t - Du_{xx} = 0 \tag{7}$$

(iii) Let $(x,t) \to (x_0, 0^+)$. If g is continuous at x_0 then $u(x,t) \to g(x_0)$

Proof. (i) From the assumption $T < \frac{1}{4aD}$ (i.e. $t < \frac{1}{4aD}$, for all $t \leq T$), we can find a positive ε such that $\frac{1}{4DT} - a > \frac{\varepsilon}{4DT}$ and hence $\frac{1}{4Dt} - a > \frac{\varepsilon}{4Dt}$ for all t (ϵ is thus independent of t). Recall that our candidate solution is defined as

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)g(y)\, dy = \frac{1}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4Dt}} g(y)\, dy$$

Using our assumption on the Cauchy data, we can bound the modulus of this candidate solution as

$$|u(x,t)| \leq \frac{c}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4Dt}} e^{ay^2}\, dy = \frac{c}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} e^{-\frac{z^2}{4Dt}} e^{a(x-z)^2}\, dz \tag{8}$$

To reach a bound of the form $|u(x,t)| \leq ce^{ax^2}$ we would like to

1) Move a term of the form e^{ax^2} outside the integral and

2) Reduce the remaining integral to something we can compute (or at least bound). In this case, the most natural approach seems to be to turn the integrand into a Gaussian pdf which is then easy to integrate on $\mathbb{R}$ (given that the pdf has total weight one).

Note that

$$-\frac{z^2}{4Dt} + a(x-z)^2 = -\left(\sqrt{\frac{1}{4Dt} - a}z + \frac{a}{\sqrt{\frac{1}{4Dt} - a}}x\right)^2 + \left(a + \frac{a^2}{\left(\frac{1}{4Dt} - a\right)}\right)x^2$$

Substituting this in (8), we get

$$|u(x,t)| \leq \frac{c}{\sqrt{4\pi Dt}} e^{\left(a+\frac{a^2}{\varepsilon}\right)x^2} \int_{\mathbb{R}} e^{-\left(\sqrt{\frac{1}{4Dt}-a}z + \frac{a}{\sqrt{\frac{1}{4Dt}-a}}x\right)^2} dz$$

We then use our time constraint, $\frac{1}{4Dt} - a > \frac{\varepsilon}{4Dt}$, which gives

$$\sqrt{\frac{1}{4Dt} - a}\frac{1}{\sqrt{\varepsilon}} > \frac{1}{\sqrt{4Dt}}$$

and hence

$$|u(x,t)| \leq \sqrt{\frac{1}{4Dt} - a}\frac{c}{\sqrt{\varepsilon}} e^{\left(a+\frac{a^2}{\varepsilon}\right)x^2} \int_{\mathbb{R}} e^{-\left(\sqrt{\frac{1}{4Dt}-a}z + \frac{a}{\sqrt{\frac{1}{4Dt}-a}}x\right)^2} dz$$

Applying the change of variables $z \leftarrow \sqrt{\frac{1}{4Dt} - a}z$ we finally get

$$\begin{aligned}|u(x,t)| &\leq \frac{c}{\sqrt{\varepsilon}} e^{\left(a+\frac{a^2}{\varepsilon}\right)x^2} \int_{\mathbb{R}} e^{-(z+\beta x)^2} dz \\ &\leq \frac{c\sqrt{2\pi}}{\sqrt{\varepsilon}} e^{\left(a+\frac{a^2}{\varepsilon}\right)x^2}\end{aligned}$$

which concludes the proof for (i)

(ii) Again, starting from the definition of our candidate solution

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)g(y)\, dy = \frac{1}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4Dt}} g(y)\, dy$$

We want to show that $u \in C^\infty(\mathbb{R} \times (0,T])$ (meaning u has derivatives of all orders) and satisfies $u_t - Du_{xx} = 0$ in the strip $\mathbb{R} \times (0,T]$. To show that the derivatives are continuous, we need to move the derivatives inside the integral. We will rely on the following theorem

Theorem 2. *Suppose* $f : X \times [a,b] \to \mathbb{C}$ $(-\infty < a < b < \infty)$ *and* $f(\cdot,t) : X \to \mathbb{C}$ *is integrable for each* $t \in [a,b]$. *Let* $F(t) = \int_X f(x,t)d\mu(x)$. *Suppose* $\frac{\partial f}{\partial t}$ *exist and there is a* $g \in L^1(\mu)$[1] *such that* $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x)$ *for all* x, t. *Then* f *is differentiable and* $F'(t) = \int_X \frac{\partial f}{\partial t}(x,t)d\mu(x)$

[1] $g \in L^1(\mu)$ means that g satisfies $\int_{\mathbb{R}} |g| d\mu < \infty$

From Theorem 2 above, we thus need to bound the derivatives of all orders in t and x $(\partial_t^h \partial_x^k \Phi(x-y,t)g(y))$ by a function whose modulus is integrable. Those derivatives read as sums of terms of the form

$$t^{-r}|x-y|^s e^{-\frac{(x-y)^2}{4Dt}} Ae^{ay^2} \quad \text{up to multiplicative constants}$$

Note that for such terms, we have

$$t^{-r}|x-y|^s e^{-\frac{(x-y)^2}{4Dt}} Ae^{ay^2} = t^{-r}|x-y|^s e^{-\frac{x^2}{4Dt}-\frac{y^2}{4Dt}+\frac{2xy}{4Dt}} Ae^{ay^2}$$

Using $(x-by)^2 = x^2 + b^2y - 2bxy \geq 0$ which gives $2xy \leq \frac{1}{b}x^2 + by^2$, we can further write

$$t^{-r}|x-y|^s e^{-\frac{x^2}{4Dt}} e^{-\frac{y^2}{4Dt}} e^{\frac{2xy}{4Dt}} Ae^{ay^2} \tag{9}$$

$$\leq t_0^{-r}\left(R+|y|\right)^s e^{-x^2\left(\frac{1}{4Dt}-\frac{1}{b}\right)} Ae^{-\left(\frac{1}{4Dt}-a-b\right)y^2} \tag{10}$$

Recall that from our time constraint, we have $\frac{1}{4Dt} > a$. Moreover, using $|a+b| \leq |a|+|b| \leq 2\max\left(|a|,|b|\right)$ and hence $|a+b|^n \leq 2^n|a|^n + 2^n|b|^n$, and taking b small enough, we have

$$\begin{aligned}(10) \leq\ & t_0^{-r} R^s (2^{s-1}) e^{-x^2\left(\frac{1}{4Dt}-\frac{1}{b}\right)} Ae^{-\left(\frac{1}{4Dt}-a-b\right)y^2} \\ &+ t_0^{-r} 2^{s-1} e^{-x^2\left(\frac{1}{4Dt}-\frac{1}{b}\right)} |y|^s e^{-\left(\frac{1}{4Dt}-a-b\right)y^2}\end{aligned}$$

with $\left(\frac{1}{4Dt}-a-b\right) > 0$. The first term is a Gaussian which is integrable on $\mathbb{R}$. To convince yourself that the second term can be bounded by a Gaussian as well, note that

$$e^{|y|} = \sum_{k=0}^{\infty} \frac{|y|^k}{k!} \geq \frac{|y|^s}{s!}$$

hence

$$\begin{aligned}\int_{-\infty}^{\infty} |y|^s e^{-\alpha y^2}\, dy &\leq 2\int_0^{\infty} s!\ e^{y} e^{-\alpha y^2}\, dy \\ &\leq 2\int_0^{\infty} s!\ e^{-\alpha(y-\frac{1}{2\alpha})^2} e^{\frac{1}{4\alpha}}\, dy\end{aligned}$$

which again is a Gaussian.

Since all the derivatives are bounded by non negative integrable functions, we can move the differential operator inside the integral and connect (2) and (3). I.e.

$$u_t - \Delta u = \int_{\mathbb{R}} \left[\partial_t \Phi(x-y) - \Delta\Phi(x-y)\right] d(y)\, dy = 0$$

This concludes the proof of (ii)

(iii) To conclude, we want to show that when g is continuous at x_0, $u(x,t) \to g(x_0)$ when $t \to 0^+$. Note that this is equivalent to showing that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x-x_0|, t < \delta$ then $|u(x_0,t) - g(x_0)| < \varepsilon$. Since we assumed that g was continuous, we can write $\forall\ \varepsilon/2 \quad \exists\ \delta$ s.t. $|y - x_0| < \delta \Rightarrow |g(y) - g(x_0)| < \varepsilon/2$. Using this, we can express the difference $u(x_0, t) - g(x_0)$ as

$$\begin{aligned}
u(x_0, t) - g(x_0) &= \int_{|y-x_0|<\delta} \Phi(x-y,t)\,[g(y) - g(x_0)]\ dy \\
&\quad + \int_{|y-x_0|>\delta} \Phi(x-y,t)\,[g(y) - g(x_0)]\ dy \\
&\leq \frac{\varepsilon}{2} + \int_{|y-x_0|>\delta} \Phi(x-y,t)\,[g(y) - g(x_0)]\ dy
\end{aligned}$$

Where we used the fact that Φ is non-negative. From our assumption on the Cauchy data, on the other hand, we have

$$|g(y) - g(x_0)| \leq Ae^{ay^2} + Ae^{ax_0^2}$$

hence

$$\begin{aligned}
\int_{|y-x_0|>\delta} \Phi(x-y,t)\,[g(y) - g(x_0)]\ dy &\leq \int_{|y-x_0|>\delta} \Phi(x-y,t)\left(Ae^{ax_0^2} + Ae^{ay^2}\right)\ dy \\
&\leq Ae^{ax_0^2} \int_{|y-x_0|>\delta} \Phi(x-y,t)\ dy \\
&\quad + A\int_{|y-x_0|>\delta} \Phi(x-y,t)e^{ay^2}\ dy
\end{aligned}$$

For both of these terms, we can apply the following reasoning (we do it for the first term, the second one can be treated in a similar manner)

$$\int_{|y-x_0|>\delta} \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{y^2}{4Dt}}\ dy \leq \int_{|y\sqrt{4\pi Dt}-x_0|>\delta} e^{-y^2}\ dy \tag{11}$$

$$\leq \int_{|y\sqrt{4\pi Dt}|>\delta} e^{-\left(y-\left(-\frac{x_0}{\sqrt{4\pi Dt}}\right)\right)^2}\ dy \tag{12}$$

$$\leq \int_{|y|>\frac{\delta}{\sqrt{4\pi Dt}}} e^{-y^2\left(1-\frac{1}{b}\right)} e^{-\frac{x_0^2}{4\pi Dt}} e^{\frac{bx_0^2}{\sqrt{4\pi Dt}}}\ dy \tag{13}$$

In (11) we use the change of variable $y \leftarrow y/\sqrt{4Dt}$. In (11), we use the change of variable $y \leftarrow y - \frac{x_0}{\sqrt{4\pi Dt}}$. In the last line we again use $2xy \leq \frac{1}{b}y^2 + bx^2$ and take b sufficiently small to satisfy $1 - \frac{1}{b} > 0$.

Taking the limit $t \to 0^+$ in (13) and noting that $e^{-\frac{a}{t}+\frac{b}{\sqrt{t}}} = e^{-\frac{a}{\sqrt{t}}\left(\frac{1}{\sqrt{t}-\frac{a}{b}}\right)}$ gives the conclusion.

$\square$

Non homogeneous problem and Duhamel's principle

We now discuss how to solve the general (non-homogeneous) Cauchy problem

$$\begin{cases} u_t - Du_{xx} = f(x,t) & \text{in } \mathbb{R} \times (0,T] \\ u(x,0) = g(x) & \text{in } \mathbb{R} \end{cases} \tag{14}$$

We will start by considering the problem

$$\begin{cases} u_t - Du_{xx} = f(x,t) & \text{in } \mathbb{R} \times (0,T) \\ u(x,0) = 0 \end{cases} \tag{15}$$

To solve this problem, we will rely on the following two steps approach known as Duhamel's principle:

1) Construct a family of solutions of homogeneous Cauchy problems with variable initial time s, $0 \leq s \leq t$ and initial data $f(x,s)$

2) Integrate the above family with respect to s over $(0,t)$ to get the solution to (15).

As an illustration of the method, consider the family of homogeneous Cauchy problems

$$\begin{cases} w_t - Dw_{xx} = 0 & x \in \mathbb{R},\ t > s \\ w(x,s) = f(x,s) & x \in \mathbb{R} \end{cases} \tag{16}$$

(here s is thus viewed as a parameter). Recall that $\Phi(x,t)$ is used to denote the fundamental solution of the heat equation with initial data $\delta(x)$ (i.e. $\Phi(x,0) = \delta(x)$) then $\Phi(x,t-s)$ is the fundamental solution that satisfies the Cauchy problem

$$\begin{cases} u_t - D\Delta u = 0 \\ u(x,s) = \delta(x) \end{cases} \tag{17}$$

In particular for general Cauchy data of the form $u(x,s) = f(x,s)$ we have

$$w(x,t,s) = \int_{\mathbb{R}} \Phi(x-y,t-s) f(y,s)\, dy \tag{18}$$

Integrating over $(0,t)$, we get

$$v(x,t) = \int_0^t w(x,t,s)\, ds = \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s) f(y,s)\, dy\, ds \tag{19}$$

We want to show that this candidate solution satisfies (15). Obviously $v(x,t)$ satisfies $v(x,0) = 0$. To prove that it also satisfies the non-homogeneous heat equation with $f(x,t)$ as the source term, we first compute the time derivative $\partial_t v$. Note that this derivative arises from two contributions (as t and s are treated as distinct parameters): a direct derivative with respect to time (that can be moved inside the integral) and the derivative with respect to time of the integral taken at $s = t$. We thus have

$$\partial_t \left(\int_0^t w(x,t,s)\, ds \right) = \int_0^t \partial_t w(x,t,s)\, ds + \left(\partial_t \int_0^t w(x,t',s)\, ds \right)\Bigg|_{t'=t}$$

Plugging this into the heat equation

$$\begin{aligned} v_t - Dv_{xx} &= w(x,t;t) + \int_0^t \left(\partial_t w(x,t,s)\ ds - Dw_{xx}(x,t,s)\right)\ ds \\ &= f(x,t) \end{aligned}$$

$v(x,t)$ is thus a valid solution of (15).

To obtain the solution for the general Cauchy problem, we use superposition. Combining the solution of the Cauchy problem for the homogeneous equation and the solution of the Cauchy problem with a forcing term $f(x,t)$ (heterogeneous heat equation) but homogeneous Cauchy data.

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) g(y)\ dy + \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s) f(y,s)\ dy\ ds \tag{20}$$

Clearly this solution satisfies our initial conditions as we have $u(x,t) \to g(x_0)$ as $(x,t) \to (x_0, 0)$. Moreover, note that when applying our differential operator to the first term in (20), the result vanishes as $\Phi(\boldsymbol{x})$ satisfies the homogeneous heat equation.

Uniqueness

So far we have discussed existence of a solution for the Cauchy problem but we haven't proved uniqueness of the solution. Proving uniqueness is relatively straightforward provided that we once again remain within the class of functions with growth at infinity controlled by an exponential of the type Ce^{Ax^2} for all $t > 0$ (such a class of functions is known as the Tychonov class). The uniqueness of the Cauchy problem can then be derived as a consequence of the following maximum principle

Theorem 3 (Global Maximum Principle)**.** *Suppose $u \in C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves*

$$\begin{cases} u_t - D\Delta u & \text{on } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{21}$$

and satisfies the growth estimates

$$|u(x,t)| \leq Ae^{a|x|^2}, \quad (x \in \mathbb{R}^n, 0 \leq t \leq T) \tag{22}$$

Moreover, assume $t < T < \frac{1}{4aD}$ for some constants $A, a > 0$ then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g$$

Proof. From the assumption on t, we have $T < \frac{1}{4aD}$ hence $\exists \varepsilon > 0$ such that $4a(T+\varepsilon)D < 1$. Let us fix $y \in \mathbb{R}$, $\mu > 0$. We consider the function $v(x,t)$ defined as

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2} D^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}}, \quad (x \in \mathbb{R}^n, t > 0)$$

Note that

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\mu}{(T+\varepsilon-t)^{n/2} D^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}}\right) \\ &= \frac{n}{2}\frac{\mu}{(T+\varepsilon-t)^{n/2+1} D^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)D}} + \frac{\mu}{(T+\varepsilon-t)^{n/2}} \frac{|x-y|^2}{4D(T+\varepsilon-t)^2} e^{\frac{|x-y|^2}{4D(T+\varepsilon-t)}} \end{aligned}$$

Moreover

$$\partial_{x_i} = \frac{\mu}{(T+\varepsilon-t)^{n/2} D^{n/2}} \frac{-2(x_i-y_i)}{4D(T+\varepsilon-t)} e^{\frac{|x-y|^2}{4D(T+\varepsilon-t)}}$$

as well as

$$\partial_{x_i x_i} = -\frac{\mu}{(T+\varepsilon-t)^{n/2}} \frac{2}{4D(T+\varepsilon-t)} e^{\frac{|x-y|^2}{4D(T+\varepsilon-t)}} + \frac{\mu}{(T+\varepsilon-t)^{n/2}} \frac{4(x_i-y_i)^2}{16D^2(T+\varepsilon-t)^2} e^{\frac{|x-y|^2}{4D(T+\varepsilon-t)}}$$

From this, we see that $v(x,t)$ satisfies $\partial_t v - \Delta v = 0$. Let $U = B(y,r)$ the ball of radius r centered on y. Recall that from the weak maximum principle, we have

$$\max_{\overline{Q}_T} v = \max_{\partial Q_T} v = \max_{|x-y|=r \cup \{t=0\}} v(x,t) \tag{23}$$

Now on the $\{t=0\}$ part of the boundary we have

$$\begin{aligned} v(x,0) &= u(x,0) - \frac{\mu}{(T+\varepsilon)^{n/2} D^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon)D}} && (24) \\ &\le u(x,0) = g(x) && (25) \end{aligned}$$

On the $|x-y| = r$ part of the boundary, we have

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2} D^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)D}} && (26) \\ &\le Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2} D^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)D}} && (27) \\ &\le Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2} D^{n/2}} e^{\frac{r^2}{4(T+\varepsilon)D}} && (28) \end{aligned}$$

In the last line, we use the fact that $\frac{1}{x^n}e^{1/x}$ is a decreasing function of x. I.e,

$$\frac{d}{dx}\left(\frac{1}{x^n}e^{1/x}\right) = -\frac{n}{x^{n+1}}e^{1/x} + \frac{1}{x^n}\left(-\frac{1}{x^2}\right)e^{1/x}$$

From the condition $T < \frac{1}{4Da}$ and $4a(T+\varepsilon) < 1$, we can always find $\gamma > 0$ such that

$$\frac{1}{4(T+\varepsilon)D} = a + \gamma$$

Substituting this in (28), we get

$$\begin{aligned} v(x,t) &\leq Ae^{a(|y|+r)^2} - \mu\left(4(a+\gamma)\right)^{n/2} e^{(a+\gamma)r^2} \\ &\leq e^{(a+\gamma)r^2}\left(-\mu(4(a+\gamma))^{n/2} + Ae^{-\gamma r^2+2ar|y|+a|y|^2}\right) \end{aligned}$$

Such an expression in particular shows that we can always take r large enough so as to satisfy

$$v(x,t) \leq e^{(a+\gamma)r^2}\left(-\mu(4(a+\gamma))^{n/2} + Ae^{-\gamma r^2+2ar|y|+a|y|^2}\right) \leq \sup_{\mathbb{R}} g$$

This implies

$$v(x,t) \leq \sup\ g \quad \text{on } |x-y| = r \tag{29}$$

Grouping (25) and (29), and substituting in (23), we finally get

$$\max_{\overline{Q}_T} v \leq \sup_{\mathbb{R}} g(x) \tag{30}$$

since y was arbitrary, we get

$$\sup_{\mathbb{R}^n\times[0,T]} v(y,t) \leq \sup\ g$$

Finally taking $\mu \to 0$ gives the conclusion

$$u(x,t) \leq \sup\ g$$

□

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