MATH-UA 9263 - Partial Differential Equations Recitation 3: Maximum Principle, Cauchy problem

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Question 1 (Salsa) *Let $g(t) = M$ for $0 \leq t \leq 1$ and $g(t) = M - (1-t)^4$ for $1 < t \leq 2$. Let u be the solution of $u_t - u_{xx} = 0$ in $Q_2 = (0,2) \times (0,2)$, $u = g$ on $\partial_p Q_2$. Compute $u(1,1)$ and check that it is the maximum of u. Is this in contrast with the maximum principle?*

Question 2 (Evans) *Write down an explicit formula for a solution of*

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Question 3 *Generalize the energy method introduced in the lecture, show that the initial Dirichlet, Neumann, Robin and mixed problems have at most one solution in $C^{2,1}(\overline{Q}_T)$. Assume $u = 0$ for the Dirichlet condition,*

Question 4 *Show that the maximum principle continues to hold for equations of the form $u_t - \nabla \cdot (A\nabla u) + b(\nabla u)$ where A is symmetric and positive definite and the function b satisfies $b(0) = 0$*

Question 5 (Salsa) *Find the similarity solutions of the equation $u_t - u_{xx} = 0$ of the form $u(x,t) = U(x/\sqrt{t})$ and express the result in terms of the error function*

$$\text{erf}(x) = \frac{2}{\sqrt{x}} \int_0^x e^{-z^2}\, dz$$

Then find the solution of $u_t - u_{xx} = 0$ in $x > 0, t > 0$ satisfying the conditions $u(0,t) = 1$, $u(x,t) \to 0$ as $x \to +\infty$, $t > 0$, and $u(x,0) = 0$, $x > 0$.

Question 6 (Strauss) *Prove the* comparison principle *for the diffusion equation: if u and v are two solutions, and if $u \leq v$ for $t = 0$, for $x = 0$, and for $x = \ell$, then $u \leq v$ for $0 \leq t < \infty$, $0 \leq x \leq \ell$.*

Question 7 (Strauss) *Consider the diffusion equation on* $(0,\ell)$ *with the Robin boundary conditions* $u_x(0,t) - a_0 u(0,t) = 0$ *and* $u_x(\ell,t) + a_\ell u(\ell,t) = 0$. *If* $a_0 > 0$ *and* $a_\ell > 0$, *use the energy method to show that the endpoints contribute to the decrease of* $\int_0^\ell u^2(x,t)\,dx$ *(This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "radiating" or "dissipative")*

Question 8 (Salsa) *Determine for which* α *and* β *there exist similarity solutions to* $u_t - u_{xx} = f(x)$ *of the form* $t^\alpha U(x/t^\beta)$ *in each one of the following cases*

$$\text{(a)}\quad f(x) = 0, \qquad \text{(b)}\quad f(x) = 1, \qquad \text{(c)}\quad f(x) = x$$

Question 9 *Find a similarity solution for the equation*

$$v_t = v_{xx} + (v^2)_x$$

Consider a solution of the form $v(x,t) = t^\alpha w(\frac{x}{t^\beta})$

a) Find the parameters α *and* β

b) Then find the ODE for $W(y)$ *and show that this ODE can be reduced to first order.*

c) Finally find the resulting solution for the ODE.

Question 10 *Solve the diffusion equation*

$$\begin{cases} u_t = D u_{xx}, & (-\infty < x < \infty, 0 < t < \infty) \\ u(x,0) = \varphi(x) \end{cases}$$

if $\varphi(x) = e^{-x}$ *for* $x > 0$ *and* $\varphi(x) = 0$ *for* $x < 0$

Question 11 (Strauss) *Compute* $\int_0^\infty e^{-x^2}\,dx$ *[Hint: this is a function that cannot be integrated by formula. So use the following trick. Transform the double integral* $\int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy$ *into polar coordinates and you'll end up with a function that can be integrated easily]*

Question 12 *Solve the diffusion equation* $u_t = k u_{xx}$ *with the initial conditions* $u(x,0) = x^2$ *by the following special method. First show that* u_{xxx} *satisfies the diffusion equation with zero initial condition. Therefore, by uniqueness ,* $u_{xxx} = 0$. *Integrating this result thrice, obtain* $u(x,t) = A(t)x^2 + B(t)x + C(t)$. *Finally, it's easy to solve for* A, B *and* C *by plugging into the original problem.*

Question 13 *(a) Now solve the equation* $u_t = k u_{xx}$ *with the initial conditions* $u(x,0) = x^2$ *by using the general formula discussed in the text. This expresses* $u(x,t)$ *as a certain integral. Substitute* $p = (x-y)/\sqrt{4kt}$ *in this integral.*

(b) *Since the solution is unique, the resulting formula must agree with the answer to question 13 above. Deduce the value of*

$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp$$

Question 14 (Strauss) *(a) Consider the diffusion equation on the whole line with the usual initial condition* $u(x,0) = \varphi(x)$*. If* $\varphi(x)$ *is an* odd *function, show that the solution* $u(x,t)$ *is also an* odd *function of* x*. [Hint: Consider* $u(-x,t) + u(x,t)$ *and use the uniqueness]*

(b) Show that the same is true if "odd" is replaced by "even"

Question 15 (Strauss) *Solve the diffusion equation with constant dissipation*

$$u_t - ku_{xx} + bu = 0, \quad \text{for } -\infty < x < \infty, \quad \text{with } u(x,0) = \varphi(x),$$

where $b > 0$ *is a constant (Hint: make the change of variables* $u(x,t) = e^{-bt}v(x,t)$*)*