MATH-UA 9263 - Partial Differential Equations
Recitation 1: general intro + diffusion

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Question 1 (Vasy, Evans) *Classify the following PDEs by degree of non-linearity (linear,semilinear, quasilinear, fully non linear)*

(i) $(\cos x)u_x + u_y = u^2$

(ii) $uu_{tt} = u_{xx}$

(iii) $u_x - e^x u_y = \cos x$

(iv) $u_{tt} - u_{xx} + e^u u_x = 0$

(v) $|Du| = 1$

(vi) $\text{div}\left(|Du|^{p-2}Du\right) = 0$

(vii) $u_t - \Delta(u^\gamma) = 0$

(viii) $u_t - \sum_{ij=1}^{n} \left(a^{ij}u\right)_{x_i,x_j} - \sum_{i=1}^{n} \left(b^i u\right)_{x_i} = 0$

(ix) $u_t - \sum_{i=1}^{n} \left(b^i u\right)_{x_i} = 0$

(x) $\text{div}\left(\frac{Du}{\left(1+|Du|^2\right)^{1/2}}\right) = 0$

Solution 1

(i) *semilinear*

(ii) *quasilinear*

(iii) *linear*

(iv) *semilinear*

(v) *non linear*

(vi) *quasi linear*

(vii) *quasi-linear*

(viii) *linear*

(ix) *linear*

(x) *quasi-linear*

Note that for (vi) *we have*

$$\begin{aligned}\operatorname{div}\left(Du|Du|^{p-2}\right) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}|Du|^{p-2}\right) \\ &= \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}\|\nabla u\|^{p-2} \\ &+ \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(\|\nabla u\|^{p-2}\right) \\ &= \Delta u \|\nabla u\|^{p-2} \\ &+ \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(p-2)\|\nabla u\|^{p-4}\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_k \partial x_i}\end{aligned}$$

For (x), *a similar reasoning yields*

$$\begin{aligned}\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right) &= \sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^2}\frac{1}{(1+|Du|^2)^{1/2}}\right) \\ &- \sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{1}{(1+|Du|^2)^{3/2}}\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_k \partial x_i}\end{aligned}$$

Question 2 (Vasy 2.1) *Recall that if* $\Omega \subset \mathbb{R}^3$ *is a domain and* $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ *is* $\mathcal{C}^1$*, the the divergence of* u *is the function* $\nabla \cdot u : \Omega \to \mathbb{R}$

$$\nabla \cdot u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3,$$

While the curl of u *is the vector field* $\nabla \times u : \Omega \to \mathbb{R}^3$,

$$\nabla \times u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)$$

Further, for $\mathcal{C}^2$ *functions* u*,* $\nabla^2 u : \Omega \to \mathbb{R}^3$ *is the vector field*

$$\begin{aligned}\nabla^2 u &= \left(\nabla^2 u_1, \nabla^2 u_2, \nabla^2 u_3\right) \\ &= \left((\partial_1^2+\partial_2^2+\partial_3^2)u_1, (\partial_1^2+\partial_2^2+\partial_3^2)u_2, \left(\partial_1^2+\partial_2^2+\partial_3^2\right)u_3\right)\end{aligned}$$

Show that

$$\nabla \times (\nabla \times u) = \nabla(\nabla \cdot u) - \nabla^2 u$$

Solution 2 *For any vector field* $\boldsymbol{u} \in \mathbb{R}^3$*, the curl can be computed as the following determinant*

$$\begin{aligned}\nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} &= \boldsymbol{i}(\partial_{x_2}u_3 - \partial_{x_3}u_2) - \boldsymbol{j}(\partial_{x_1}u_3 - \partial_{x_3}u_1) \\ &+ \boldsymbol{k}(\partial_{x_1}u_2 - \partial_{x_2}u_1)\end{aligned}$$

Using this, we get

$$\begin{aligned}\nabla\times(\nabla\times\boldsymbol{u}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ (\partial_{x_2}u_3-\partial_{x_3}u_2) & (\partial_{x_3}u_1-\partial_{x_1}u_3) & (\partial_{x_1}u_2-\partial_{x_2}u_1) \end{vmatrix} \\ &= \begin{bmatrix} \partial_{x_2}\partial_{x_1}u_2-\partial_{x_2}^2u_1-\partial_{x_3}^2u_1+\partial_{x_1}\partial_{x_3}u_3 \\ -\partial_{x_1}^2u_2+\partial_{x_1}\partial_{x_2}u_1+\partial_{x_3}\partial_{x_2}u_3-\partial_{x_3}^2u_2 \\ -\partial_{x_2}^2u_3+\partial_{x_2}\partial_{x_3}u_2+\partial_{x_1}\partial_{x_3}u_1-\partial_{x_1}^2u_3 \end{bmatrix} \qquad (1)\end{aligned}$$

On the other hand

$$\nabla(\nabla\cdot\boldsymbol{u}) = \begin{bmatrix} \partial_{x_1}^2u_1+\partial_{x_1}\partial_{x_2}u_2+\partial_{x_1}\partial_{x_2}u_3 \\ \partial_{x_3}\partial_{x_1}u_1+\partial_{x_3}\partial_{x_2}u_2+\partial_{x_3}^2u_3 \\ \partial_{x_3}\partial_{x_1}u_1+\partial_{x_3}\partial_{x_2}u_2+\partial_{x_3}^2u_3 \end{bmatrix}$$

Hence

$$\nabla(\nabla\cdot\boldsymbol{u})-\nabla^2\boldsymbol{u} = \begin{bmatrix} \partial_{x_1}^2u_1+\partial_{x_1}\partial_{x_2}u_2+\partial_{x_1}\partial_{x_3}u_3 \\ \partial_{x_2}\partial_{x_1}u_1+\partial_{x_2}^2u_2+\partial_{x_2}\partial_{x_3}u_3 \\ \partial_{x_3}\partial_{x_1}u_1+\partial_{x_3}\partial_{x_2}u_2+\partial_{x_3}^2u_3 \end{bmatrix} - \begin{bmatrix} \partial_{x_1}^2u_1+\partial_{x_2}^2u_1+\partial_{x_3}^2u_1 \\ \partial_{x_1}^2u_2+\partial_{x_2}^2u_2+\partial_{x_3}^2u_2 \\ \partial_{x_1}^2u_3+\partial_{x_2}^2u_3+\partial_{x_3}^2u_3 \end{bmatrix}$$

from which we recover (1)

Question 3 *Find the distribution of temperature in a long cylindrical tube with inner radius $r = a$ and outer radius $r = b$ which is maintained at temperature $u(r = a) = T_a$ and $u(r = b) = T_b$. (Stationnary case without heat generation)*

Solution 3 *We first recall how to derive the expression of the Laplacian in cylindrical coordinates. We let*

$$\begin{aligned} x_1 &= r\cos\theta \\ x_2 &= r\sin\theta \\ x_3 &= z \end{aligned}$$

Viewing $u(x_1, x_2, x_3) = u(r\cos\theta, r\sin\theta, z)$ as a function of r, θ and z, we get

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_1}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x_1} \\ &= \frac{\partial u}{\partial r}\frac{\partial x_1}{\partial r}+\frac{\partial u}{\partial\theta}\frac{-x_2}{r^2}\end{aligned}$$

Similarly, we have

$$\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial r}\frac{x_2}{r}+\frac{\partial u}{\partial\theta}\frac{x_1}{r^2}$$

Using $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, we finally

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \frac{\partial u}{\partial r}\cos\theta+\frac{\partial u}{\partial\theta}\frac{(-\sin\theta)}{r} \\ \frac{\partial u}{\partial x_2} &= \frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}\end{aligned}$$

From this, the Laplacian is given by

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) + \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

Developing each of those terms, we have

$$\begin{aligned} &\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial r}\right) \\ &= \cos^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial \theta \partial r} \\ &+ \frac{\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} \\ &+ \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \end{aligned}$$

as well as

$$\begin{aligned} &\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right) \\ &= \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial \theta \partial r} \\ &- \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} \\ &- \frac{\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \end{aligned}$$

Combining those two expressions, we get

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Question 4 *We consider the heat equation in cylindrical coordinates*

$$\frac{\partial u}{\partial t} = \frac{k_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad a < r < b \tag{2}$$

(i) *Derive the equation above from the Laplacian and the change of variables*

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

(ii) *We consider equation (2) subject to the following conditions*

$$u(r,0)=f(r),\ \frac{\partial u}{\partial r}(a,t)=\beta,\ \frac{\partial u}{\partial r}(b,t)=1$$

Using physical reasoning, for what value(s) of β does an equilibrium temperature distribution exist?

Solution 4

(i) *see solution 3*

(ii) *Consider the steady state, we have*

$$0=\frac{\partial u}{\partial t}=\frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

From the steady state equation, we derive

$$u(r)=A\log r+B$$

Using the boundary conditions, we get

$$\frac{\partial u}{\partial r}(b)=1\ \Rightarrow\ c=b$$
$$\frac{\partial u}{\partial r}(a)=\beta\Rightarrow\frac{b}{a}=\beta$$

Question 5 *Find the temperature profile (steady state) in a long and thin wall with the following parameters: $u(x=0)=1000°C=T_1$, $u(x=L)=200°C=T_2$, $L=1m$, $k_0=1W/m°C$. Same question if the conductivity varies as k_0+k_1u.*

Solution 5

(i) *Viewing the wall from the top, we have a problem similar to the rod. The wall satisfies the system*

$$\begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{K_0}{c\rho}\dfrac{\partial^2 u}{\partial x^2}\\ \\ u(0)=1000°,\ u(L)=200°\end{cases}$$

The steady state gives $u(x)=Ax+B$ and the boundary conditions give

$$u(x,t=\infty)=\frac{-800x}{L}+1000$$

(ii) *For the variable conductivity, combining the conservation of energy with Fourier's law, we get*

$$\phi = -(K_0 + K_1 u)\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = \frac{1}{\rho c}\frac{\partial}{\partial x}\left((k_0 + k_1 u)\frac{\partial u}{\partial x}\right)$$

Looking at the steady state, integrating a first time with respect to x, we get

$$(k_0 + k_1 u)\frac{\partial u}{\partial x} = cst$$

letting $v(x) = k_0 + k_1 x$ we get

$$v(x)\frac{1}{k_1}\frac{\partial v}{\partial x} = cst$$

From which we get the steady state solution

$$v^2(x) = Ax + B$$

since the conductivity is positive, we take

$$k_0 + k_1 u(x) = \sqrt{Ax + B} \;\Rightarrow u(x) = -\frac{k_0}{k_1} + \frac{1}{k_1}\sqrt{Ax + B}$$

The solution follows from applying the boundary conditions.

$$u(0) = -\frac{k_0}{k_1} + \frac{\sqrt{B}}{k_1} = T_1 = 1000^\circ$$
$$u(L) = -\frac{k_0}{k_1} + \frac{\sqrt{AL + B}}{k_1} = T_2 = 200^\circ$$

Together we finally obtain

$$B = (k_1 T_1 + k_0)^2$$
$$A = \frac{1}{L}\left[(k_1 T_2 + k_0)^2 - (k_1 T_1 + k_0)^2\right]$$

Question 6 *Consider a uniform circular disk whose entire surface is insulated. Assume that its temperature at $t = 0$ is a function only of the distance r from the center of the disk. Starting from the basic physical laws, show that the temperature $u(r, t)$ of the disk satisfies the equation*

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{3}$$

Solution 6 *To write down the conservation of heat for the disk, we consider a small annulus of thickness dr located between $r = a$ and $r = b$. Since the heat radiates all over the surface of the disk, it makes sense to write down the conservation of energy for an annulus. In such a small annulus, if we let $e(r,t)$ denote the energy per unit area (and assuming no source of heat inside the annulus), we have*

$$\frac{\partial}{\partial t}e(r,t)Adr = \frac{\partial}{\partial t}e(r,t)2\pi r dr = (\phi(r,t)2\pi r - \phi(r+dr,t)2\pi(r+dr))$$

Using Fourier's law, expressing the heat from the temperature as $e(r,t) = c\rho u(r,t)$ and taking $dr \to 0$, we further get

$$\begin{aligned} c\rho\frac{\partial}{\partial t}u(r,t)2\pi r &= -\frac{\partial}{\partial r}\left(\phi(r,t)2\pi r\right) \\ &= K_0\frac{\partial}{\partial r}\left(2\pi r\frac{\partial u}{\partial r}\right) \end{aligned}$$

From which one can conclude using the product rule

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

Question 7 *Consider the flow of heat in a long thin bar (one dimensional case). If the bar gains heat due to radioactive decay, determine the equation satisfied by the temperature $u(x,t)$. Assume that the rate at which heat is gained per unit volume by radioactive decay is proportional to e^{-ax}.*

Solution 7 *From the radioactive decay, assuming a cross sectional area A, we get*

$$\frac{\partial e(x,t)}{\partial t}Adx = \alpha e^{-\beta x}Adx + A\left(\phi(x,t) - \phi(x+dx,t)\right)$$

Using Fourier's law and expressing the heat energy from the temperature $u(x,t)$, we finally get

$$\frac{\partial u}{\partial t} = \frac{\alpha}{\rho c}e^{-\beta x} + k\frac{\partial^2 u}{\partial x^2}$$

Question 8 *A thin bar is conducting heat along its length and is also radiating heat from its surface. The temperature of the surroundings is U. Assume that the radiation obeys Newton's law, that is, the time rate of change of temperature due to radiation is proportional to the temperature difference between the body and its surroundings. The bar has cross-sectional area A and a density δ. Derive the differential equation satisfied by the temperature.*

Solution 8 *From Newton's law, we have*

$$\frac{\partial u}{\partial t} = (u - T)\alpha$$

From this we derive

$$\rho c \frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2} + (u - T)\alpha\rho c$$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \alpha(u - T)$$

Question 9 *Consider the one dimensional heat equation in the case where heat is gained by the decomposition of the material in the bar. Derive the differential equation satisfied by the temperature if the rate at which heat is gained per unit volume by decomposition is a constant, A.*

Solution 9 *The reasoning is the same as for question 7. For a cross sectional area S, we have*

$$\frac{\partial e}{\partial t} S dx = ASdx + [\phi(x,t) - phi(x + dx, t)]S$$

hence, using Fourier's law

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho} A + \frac{k_0}{c\rho} \frac{\partial^2 u}{\partial x^2}$$

Question 10 *Assume that the temperature is spherically symmetric, $u = u(r,t)$, where r is the distance from a fixed point ($r^2 = x^2 + y^2 + z^2$). Consider the heat flow (without sources) between any two concentric spheres or radii a and b.*

1. *Show that the total heat energy is $4\pi \int_a^b c\rho u r^2 \, dr$*
2. *Show that the flow of heat energy per unit time out of the spherical shell at $r = b$ is $-4\pi b^2 K_0 \partial u/\partial r\big|_{r=b}$. A similar result holds at $r = a$.*
3. *Use parts* (i) and (ii) to derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right)$$

Solution 10

1. *Recall that the surface element of the sphere (see 1) is given by*

$$dS = r^2 \sin\phi \, d\theta \, d\phi$$

From this we get the surface of sphere of radius a as

$$\int_0^{\pi} \int_0^{2\pi} r^2 \sin\varphi \, d\theta \, d\varphi = 4\pi a^2$$

Similarly, the volume element is given by

$$dV = r \sin\varphi \, d\theta \; r \, d\varphi \, dr$$

We can then express the conservation of energy by integrating over a thin shell

$$\int_0^{2\pi} \int_0^{\pi} \int_{r=a}^{r=b} e(r,t) r^2 \sin\varphi \, d\theta \, d\theta \, d\varphi \, dr$$

Using $e(r,t) = c\rho u(r,t)$, and integrating over φ and θ, we get

$$\int_{r=a}^{r=b} c\rho u(r,t) r^2 4\pi dr$$

2. *The flux is directed in the direction normal to the constant temperature surfaces. Hence $\phi = -K\frac{\partial u}{\partial r}$. The total flux leaving the surface of the sphere of radius $r = b$ can be derived as*

$$\int_0^{2\pi} \int_0^{\pi} -K\frac{\partial u}{\partial r} r^2 \sin\varphi \, d\varphi \, d\theta$$
$$= -4\pi K b^2 \left.\frac{\partial u}{\partial r}\right|_{r=b}$$

3. *As we did with the disk, using the conservation of energy inside the $[a,b]$ shell, we have*

$$\int_{r=a}^{r=b} c\rho u(r,t) 4\pi r^2 \, dr = -4\pi K_0 \left.\frac{\partial u}{\partial r}\right|_{r=b} b^2 + 4\pi K_0 \left.\frac{\partial u}{\partial r}\right|_{r=a} a^2$$
$$- 4\pi K_0 \int_{r=a}^{r=b} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) \, dr$$

Since the integral holds for any radii a and b, by symmetry, we get

$$\frac{\partial u}{\partial t} = \frac{K_0}{c\rho} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$$

Question 11 *Determine the steady state temperature distribution between two concentric spheres of radii* 1 *and* 4 *respectively, if the outer sphere is maintained at* 80° *and the inner sphere is maintained at* 0°

Solution 11 *Using* (3)*, we get*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)\dfrac{K_0}{\rho c} \\ \\ u(1,t) = 0°,\ u(4,t) = 80° \end{cases} \tag{4}$$

The steady equation then gives

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = 0$$

From this we get

$$r^2\frac{\partial u}{\partial r} = cst \Rightarrow \frac{\partial u}{\partial r} = \frac{c}{r^2}$$

Hence

$$u(r) = -\frac{A}{r} + B$$

Using the boundary conditions, we finally obtain

$$u(r) = -\frac{320}{3r} + \frac{320}{3}$$

References

[1] Jean-François Remacle, Grégoire Winckelmans, *FSAB1103 - Mathématiques 3/Équations aux Dérivées Partielles*, 2007.

[2] Richard Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Fourth Edition, Pearson 2004.

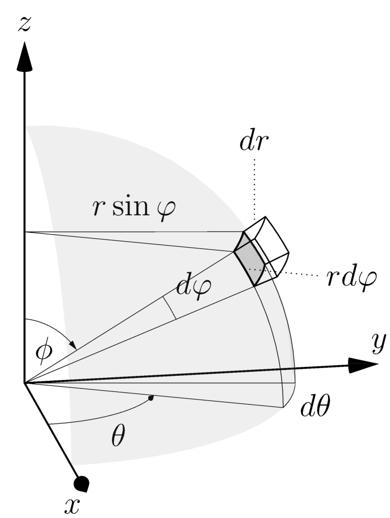

Figure 1: Spherical surface and volume elements