Introduction to Optimisation, Lecture 3

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Duality

Consider the linear program

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & 4x_1 + 8x_2 \leq 12 \\ & 2x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{1}$$

Without computing the optimum, we can infer from the first inequality and from the non negativity constraints that the objective is not larger than 12. I.e.

$$\forall \quad x_1, x_2 \geq 0, \quad (2x_1 + 3x_2) \leq 4x_1 + 8x_2 \leq 12 \tag{2}$$

We can in fact obtain a better upper bound if we divide the first inequality by two. I.e.

$$2x_1 + 3x_2 \leq \frac{1}{2}(4x_1 + 8x_2) \leq \frac{12}{2}$$

An even better bound can be derived if we add the first two inequalities and divide by 3. I.e.

$$2x_1 + 3x_2 \leq \frac{1}{3}\left[(4x_1 + 8x_2) + (2x_1 + x_2)\right] \leq \frac{1}{3}15 = 5 \tag{3}$$

From this, we in particular see that the objective cannot be larger than 5.

How good an upper bound can we get in this way? If we look at what we have just done, the idea we followed was that we tried to derive an inequality of the form

$$d_1 x_1 + d_2 x_2 \leq h \tag{4}$$

where $d_1 \geq 2$, $d_2 \geq 3$ and h is as small as possible (so as to generate a bound that is as precise as possible on the objective). For such choices of d_1 and d_2, we then have

$$2x_1 + 3x_2 \leq d_1 x_1 + d_2 x_2 \leq h \tag{5}$$

Let us try to formalize what we have learned and let us look for the best combination of our constraints so that our upper bound on the objective is minimized (i.e. as accurate as possible). For any non negative y_1, y_2 and y_3, we can combine the constraints and generate the additional inequality

$$y_1 (4x_1 + 8x_2) + y_2 (2x_1 + x_2) + y_3 (3x_1 + 2x_2) \leq 12y_1 + 3y_2 + 4y_3 \tag{6}$$

Or equivalently,

$$(4y_1 + 2y_2 + 3y_3)\, x_1 + (8y_1 + y_2 + 2y_3)\, x_2 \leq 12y_1 + 3y_2 + 4y_3 \tag{7}$$

In particular, to obtain an upper bound on the objective, we need to choose y_1, y_2, y_3 such that

$$4y_1 + 2y_2 + 3y_3 \geq 2 \tag{8}$$

$$8y_1 + y_2 + 2y_3 \geq 3 \tag{9}$$

Provided that our values of y_1, y_2, y_3 satisfy the constraints (8) and (9), we can then pick the values y_1, y_2, y_3 that minimize the weighted combination $12y_1 + 3y_2 + 4y_3$ so as to get the best upper bound on the objective of our original problem. Mathematically, we can write this problem as

$$\begin{aligned} \min \quad & 12y_1 + 3y_2 + 4y_3 \\ \text{s.t.} \quad & 4y_1 + 2y_2 + 3y_3 \geq 2 \\ & 8y_1 + y_2 + 2y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \tag{10}$$

The resulting problem (10) is known as the dual of our original LP. As we saw, the dual provides an upper bound on the value of our original linear program. In this case, we can check that the solution of the dual problem is given by $y = \left(\frac{5}{16}, 0, \frac{1}{4}\right)$ with an objective value given by 4.75. One can also check that this is also the optimal value of the primal LP (1)

Let us summarize our discoveries. For a linear program of the form

$$\max \quad \boldsymbol{c}^T \boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \text{and} \quad \boldsymbol{x} \geq 0 \tag{11}$$

we can define the dual program as

$$\begin{aligned} \min \quad & \boldsymbol{b}^T \boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{A}^T \boldsymbol{y} \geq \boldsymbol{c}, \quad \boldsymbol{y} \geq 0 \end{aligned} \tag{12}$$

The following propositions known as weak and strong duality theorems characterize the relation between the primal and dual formulations (11)- (12).

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Constraint matrix	$\boldsymbol{A}$	$\boldsymbol{A}^T$
Right-hand side	$\boldsymbol{b}$	$\boldsymbol{c}$
Objective	$\max\ \boldsymbol{c}^T\boldsymbol{x}$	$\min\ \boldsymbol{b}^T\boldsymbol{y}$
Constraint direction	i^{th} constraint ≤ 0	$y_i \geq 0$
and variable sign	i^{th} constraint ≥ 0	$y_i \leq 0$
	i^{th} constraint $= 0$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	j^{th} constraint ≥ 0
	$x_j \leq 0$	j^{th} constraint ≤ 0
	$x_j = 0$	j^{th} constraint $= 0$

Proposition 1 (Weak Duality). *For each feasible solution $\boldsymbol{y}$ of the dual linear program (12), the dual value $\boldsymbol{b}^T\boldsymbol{y}$ provides an upper bound on the maximum of the objective function of the linear program (11), In other words, for each feasible solution $\boldsymbol{x}$ of (11), and each feasible solution $\boldsymbol{y}$ of (12), we have*

$$\boldsymbol{c}^T\boldsymbol{x} \leq \boldsymbol{b}^T\boldsymbol{y} \tag{13}$$

In particular, if (11) is unbounded, (12) has to be infeasible and if (12) is unbounded (from below) then (11) is infeasible.

Proposition 2 (Strong Duality). *For the linear programs*

$$\max \quad \boldsymbol{c}^T\boldsymbol{x} \quad \textit{s.t.} \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \textit{and} \quad \boldsymbol{x} \geq 0 \tag{14}$$

and

$$\min \quad \boldsymbol{b}^T\boldsymbol{y} \quad \textit{s.t.} \quad \boldsymbol{A}^T\boldsymbol{y} \geq \boldsymbol{c}, \quad \textit{and} \quad \boldsymbol{y} \geq 0 \tag{15}$$

Exactly one of the following possibilities occurs

1. *Neither (14) nor (15) has a feasible solution*
2. *(14) is unbounded and (15) has no feasible solution*
3. *(14) has no feasible solution and (15) is unbounded*
4. *Both (14) and (15) have a feasible solution. Then both have an optimal solution, and if $\boldsymbol{x}^*$ is an optimal solution of (14) and $\boldsymbol{y}^*$ is an optimal solution of (15) then*

$$\boldsymbol{c}^T\boldsymbol{x}^* = \boldsymbol{b}^T\boldsymbol{y}^* \tag{16}$$

that is the maximum of (14) equals the minimum of (15)

Solving a linear program usually provides more information than the mere optimal value of the decision variables. To an optimal solution we can associate "shadow prices" which give an indication on how much profit a company can gain, when one of the constraints can be relaxed. We discuss this idea in more detail in the next section

Shadow prices

The shadow price associated to a particular constraint represents the change in the value of the objective function per unit increase in the right-hand side of this constraint.

As an example, let us consider the problem of maximmizing the profit of a bike manufacturer. We will assume that the company assembles two models: model 1 and model 2 on two separate assembly lines. Moreover, we will assume that the company operates a painting line, which is used for both bike models, as well as a wheel production line. The LP formulation of the problem reads as follows

$$\begin{array}{lll} \max & 2.9x_1 + 2.6x_2 & \textit{(profit in 1000 €)} \\ s.t. & 4.2x_1 \leq 100 & \textit{(capacity of bike 1 assembly line)} \\ & 5.7x_2 \leq 100 & \textit{(capacity of bike 2 assembly line)} \\ & 3.8x_1 + 2.9x_2 \leq 100 & \textit{(painting line maximum capacity)} \\ & 3.1x_1 + 5.8x_2 \leq 100 & \textit{(wheel production line max capacity)} \\ & x_1, x_2 \geq 0 & \textit{(non negative production)} \end{array} \tag{17}$$

The optimal solution of this problem (you can check it with GLPK*) is given by*

$$\begin{aligned} x_1^* &= 22.22 \\ x_2^* &= 5.36 \end{aligned} \tag{18}$$

Now let us assume that instead of owning the facilities and maximizing its profit, the company has to rent those facilities. How much rent should the company typically pay?

Obviously, prices at which the company should rent the facilities should always be non negative, $y_j \geq 0$. *Moreover, since the assembly lines each have a total capacity of 100 hours, let us write the total rent (for the four lines) as*

$$r = 100y_1 + 100y_2 + 100y_3 + 100y_4 \tag{19}$$

Clearly, the money we get for renting the assembly lines should always at least match the money we make by producing and selling our bikes through the facilities. If we let x_1^* *and* x_2^* *to denote the maximum numbers of bikes that we can produce with the business, according to the primal, the maximal profit we can generate with those bikes is given by*

$$2.9x_1^* + 2.6x_2^* \tag{20}$$

Now if we rent our assembly lines at a respective cost per hour of y_1, y_2, y_3 *and* y_4*, we must therefore have (still according to the primal)*

$$4.2x_1^*y_1 + 3.8x_1^*y_3 + 3.1x_1^*y_4 \geq 2.9x_1^* \tag{21}$$

I.e. the amount of money we get for the equivalent production time of bike 1 should be bigger than the profit we make by actually producing and selling this bike. And *similarly for the second bike*

$$5.7x_2^* y_2 + 2.9x_2^* y_3 + 5.8x_2^* y_4 \geq 2.6x_2^* \tag{22}$$

Assuming that x_1^* *and* x_2^* *do not vanish, we can thus write the renting problem as*

$$\begin{array}{ll} \min & 100y_1 + 100y_2 + 100y_3 + 100y_4 \\ s.t. & 4.2y_1 + 3.8y_3 + 3.1y_4 \geq 2.9 \\ & 5.7y_2 + 2.9y_3 + 5.8y_4 \geq 2.6 \\ & y_1, y_2, y_3, y_4 \geq 0 \end{array} \tag{23}$$

Note that had we decided to combine the two constraints and to request a rent satisfying the constraint

$$(4.2y_1 + 3.8y_3 + 3.1y_4)\, x_1^* + (5.7y_2 + 2.9y_3 + 5.8y_4)\, x_2^* \geq 2.9x_1^* + 2.6x_2^* \tag{24}$$

it would not have been possible to get rid of the unknowns x_1^* *and* x_2^**.*

So the dual can be used to derive a reasonable rent on the facilities. How about the shadow prices then?

If we go back to the primal problem, we could wonder how much a relaxation regarding the firm production constraints could increase the firm's profit.

In other words, if we for example relax the maximum capacity of the painting line by one hour, we get the problem

$$\begin{array}{lll} \max & 2.9x_1 + 2.6x_2 & \\ s.t. & 4.2x_1 & \leq 100 \\ & 5.7x_2 & \leq 100 \\ & 3.8x_1 + 2.9x_2 & \leq 101 \\ & 3.1x_1 + 5.8x_2 & \leq 100 \\ & x_1, x_2 & \geq 0 \end{array} \tag{25}$$

the solution in this case turns into

$$x_1 = 22.67$$
$$x_2 = 5.1264$$

and the profit is increased as

$$\pi = 2.9(22.67) + 2.6(5.1264) \tag{26}$$

The gain in profit is thus given by $\Delta\pi = 79.06 - 78.39$*. Such a change is known as the shadow price associated to the painting line.*

In this case, the shadow price associated to the capacity constraint on the painting line is thus given by 0.67€*. Such a shadow price indicates that it can be profitable to inject up to* 0.67€ *to increase the capacity of the painting line by one hour.*

We can apply the same procedure on the second constraint. For this constraint, our production becomes

$$x_1^* = 22$$
$$x_2^* = 5.65$$

and the profit becomes

$$\pi = 2.9(22) + 2.6(5.65) = 78.5$$

Just as before, the shadow price for this constraint is thus given by

$$\Delta\pi = 78.5 - 78.39 = .11$$

If we apply the same reasoning to the assembly lines, we get for the first assembly line

$$x_1^* = 22.22$$
$$x_2^* = 5.3639$$

$\pi = 78.39 \Rightarrow \Delta\pi = 0$.

The profit will thus not be affect by a relaxation of this constraint. The same is true for the bike 2 assembly line. This observation also confirms that the binding or tight constraints at the optimum are the painting and wheel production line constraints.

Just as we did it for the constraints, it is also possible to study the evolution of the profit as a function of the non negativity constraints. In this case the shadow prices are refered to as reduced costs. I.e. the reduced cost associated with a non negative constraint is the change in the objective function per unit increase in the lower bound on the value of the corresponding variable.

As an example, increasing the right-hand side of the constraint $x_2 \geq 0$ *by one unit to* $x_2 \geq 1$ *will force the business to produce the model 2 bike. In the case of problem* (17) *since the optimal solution is already given by a number of bikes* $x_2^* \geq 1$ *(see* (18)*), changing the constraint will have no effect on the solution.*

The reduced cost associated to a given variable can in fact also be computed from the simplex tableaux.

To see this, let us consider the following problem. We are still interested in a bike manufacturing company. But this time, we will assume that it produces 3 models of bikes (we denote the corresponding numbers as x_1*,* x_2 *and* x_3*). The total profit is assumed to be* $5x_1 + 4x_2 + 7x_3$ *and we assume that the constraints arise from a single production line (let us say as a maximum production time)*

$$7x_1 + 5x_2 + 9x_3 \leq 65 \tag{27}$$

as well as on lets say the total storage capacity of the company

$$9x_1 + 21x_2 + 9x_3 \leq 140 \tag{28}$$

Finally we restrict the production of models 1 and 2 to not exceed the following upper bounds

$$x_1 \leq 9 \tag{29}$$
$$x_2 \leq 6 \tag{30}$$

Writing the problem in standard form, we get

$$\begin{array}{llllllllll} \max & 5x_1 & +4x_2 & +7x_3 & & & & & \\ s.t. & 7x_1 & +5x_2 & +9x_3 & +x_4 & & & & = 65 \\ & 9x_1 & +21x_2 & +9x_3 & & +x_5 & & & = 140 \\ & x_1 & & & & & +x_6 & & = 9 \\ & & x_2 & & & & & +x_7 & = 9 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq 0 \end{array} \tag{31}$$

The first SIMPLEX *tableau is given by*

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
7	5	9	1	0	0	0	65
9	21	9	0	1	0	0	140
1	0	0	0	0	1	0	9
0	1	0	0	0	0	1	6
-5	-4	-7	0	0	0	0	

Starting with the most negative entry in the reduced cost vector, we get x_3 as our entering variable and x_4 as the leaving variable.

Applying the following operations on the rows

$$R_2 \leftarrow R_2 - R_1 \tag{32}$$
$$R_1 \leftarrow R_1/9 \tag{33}$$
$$R_4 \leftarrow R_4 + 7R_1 \tag{34}$$

we get the tableau

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
7/9	5/9	1	1/9	0	0	0	65/9
2	16	0	1	1	0	0	75
1	0	0	0	0	1	0	9
0	1	0	0	0	1	0	6
4/9	-1/9	7	7/9	0	0	0	$7(\frac{65}{9})$

As a result, given that the reduced cost vector still has negative entries, we thus continue using x_2 as our next leaving variable. Choosing x_5 as the entering variable, and applying the following operations on the rows,

$$R_2 \leftarrow R_2/16 \tag{35}$$
$$R_1 \leftarrow R_1 - 5/9R_2 \tag{36}$$
$$R_4 \leftarrow R_4 - R_2 \tag{37}$$

we get the tableau

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
$\frac{29}{36}$	0	1	$\frac{21}{144}$	$-\frac{5}{144}$	0	0	665/144
$\frac{2}{16}$	1	0	$-\frac{1}{16}$	$\frac{1}{16}$	0	0	75/16
1	0	0	0	0	1	0	9
$-\frac{2}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	0	1	6
$\frac{11}{24}$	0	7	$\frac{111}{144}$	$\frac{1}{144}$	0	0	

whose associated reduced cost vector is given as

$$\boldsymbol{r}^* = \left(\frac{11}{24}, 0, 7, \frac{111}{144}, \frac{1}{144}, 0, 0\right) \tag{38}$$

The final solution is given by

$$\boldsymbol{x}^* = \left(0, \frac{75}{16}, \frac{665}{144}, 0, 0, 9, 6\right) \tag{39}$$

From the reduced cost vector (38), recall that at the optimum, the change in the objective given by a small change in the decision variables, can read as

$$\Delta\pi = r^* \Delta\boldsymbol{x} \tag{40}$$

Taking $\Delta\boldsymbol{x} = (1, 0, 0, 0, 0, 0, 0)$ *we can then study the effect of forcing a non zero* x_1*. In this example*

$$\Delta\pi = \boldsymbol{r}^*\boldsymbol{e}_1 = r_1 = \frac{11}{24} \tag{41}$$

Shadow prices and duality

The shadow prices can also be related to the dual variables. To see how, first recall that only those constraints that are binding are associated to non zero shadow prices. As a result if we let $\overline{\boldsymbol{A}}$ *to denote the part of* $\boldsymbol{A}$ *corresponding to the binding constraints, as we saw the shadow prices are derived by considering the relaxation*

$$\overline{\boldsymbol{A}}\boldsymbol{x}_\Delta = \boldsymbol{b} + \boldsymbol{e}_i\Delta \tag{42}$$

Solving those relations, we get

$$\boldsymbol{x}_\Delta = \overline{\boldsymbol{A}}^{-1}(\boldsymbol{b} + \boldsymbol{e}_i\Delta) \tag{43}$$

and the corresponding change in the objective can be derived as

$$\begin{aligned} \boldsymbol{c}^T\boldsymbol{x}_\Delta - \boldsymbol{c}^T\boldsymbol{x}^* &= \boldsymbol{c}^T\overline{\boldsymbol{A}}^{-1}(\boldsymbol{b} + \boldsymbol{e}_i\Delta) - \boldsymbol{c}^T\boldsymbol{x}^* && (44)\\ &= \boldsymbol{c}^T\overline{\boldsymbol{A}}^{-1}(\boldsymbol{b} + \boldsymbol{e}_i\Delta) - \boldsymbol{c}^T\overline{\boldsymbol{A}}^{-1}\boldsymbol{b} && (45) \end{aligned}$$

The Shadow prices are given by the relative change in the objective following a relaxation of the constraint. Those prices must therefore obey

$$\begin{aligned} \boldsymbol{y}_i^* &= \frac{1}{\Delta}\left(\boldsymbol{c}^T\boldsymbol{x}_\Delta - \boldsymbol{c}^T\boldsymbol{x}^*\right) && (46)\\ &= \boldsymbol{c}^T\overline{\boldsymbol{A}}^{-1}\boldsymbol{e}_i && (47) \end{aligned}$$

Grouping those prices in a single (column) vector $\boldsymbol{y}_B$*, we can write*

$$\boldsymbol{y}_B^T = \boldsymbol{c}^T \overline{\boldsymbol{A}}^{-1} \tag{48}$$

Note that this also implies $\boldsymbol{y}_B^T \overline{\boldsymbol{A}} = \boldsymbol{c}^T$ *or equivalently* $\overline{\boldsymbol{A}}^T \boldsymbol{y}_B = \boldsymbol{c}$*. Now if we introduce the vector* $\boldsymbol{y}$ *whose entries are defined as*

$$y_i = \begin{cases} (y_B)_i & \textit{for } i \textit{ corresponding to a binding constraint} \\ \\ 0 & \textit{otherwise} \end{cases} \tag{49}$$

we can write

$$\boldsymbol{A}^T \boldsymbol{y} = \boldsymbol{c} \tag{50}$$

where $\boldsymbol{A}$ *is the full matrix of the the LP (i.e* $\overline{\boldsymbol{A}}$ *augmented with the rows corresponding to the non binding constraints). This last relation in particular shows that the vector* $\boldsymbol{y}$ *of shadow prices is dual feasible. In fact we can say more as we have*

$$\boldsymbol{c}^T \boldsymbol{x}^* = \boldsymbol{c}^T \overline{\boldsymbol{A}}^{-1} \boldsymbol{b} = \boldsymbol{y}^T \boldsymbol{b} \tag{51}$$

By the duality theorem, the vector (49) of shadow prices is not only dual feasible but also dual optimal. The shadow prices are the solutions of the dual problem!

References

[1] Pablo Pedregal, *Introduction to optimization*, volume 46, Springer 2004.

[2] *Understanding and using linear programming*, Jiří Matoušek, Bernd Gärtner, volume 33, Springer 2007.

[3] *Applied mathematical programming*, Stephen P Bradley, Arnoldo C Hax and Thomas L Magnanti Addison-Wesley 1977.