Partial Differential Equations - MATH-UA 9263
Midterm

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Total: 35 points
Total time: 1h15
General instructions: The exam consists of 3 questions (each question consisting itself of 2 or 3 subquestions). Once you are done, make sure to write your name on each page, then take a picture of all your answers and send it by email to acosse@nyu.edu. In case you have any question, you can ask those through the chat. Answer as many questions as you can starting with those you feel more confident with.

Question 1 (Heat equation, 15pts)

1. *[5pts] We want to derive the heat equation on the annular region $\mathcal{R}$ shown in Fig. 1. We assume that the distribution of temperature is radially symmetric. Using Fourier's law of heat conduction (relating the heat flux $\varphi(r)$ and the temperature $u(r,t)$),*

$$\varphi(r) = -\kappa_0 \frac{\partial u}{\partial r}$$

together with the expression of the thermal energy $e = \rho\, c\, u(r,t)$, show that the heat equation inside the annulus reads as

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} \right) \tag{1}$$

2. *[10pts] We now consider the following one dimensional problem*

$$\begin{cases} (x+1)u_t = 2u_x + (x+1)u_{xx} & 0 < x < 2 \\ u(0,t) = 1 & t > 0 \\ u(2,t) = 2 & t > 0 \\ u(x,0) = x & 0 \le x \le 2 \end{cases} \tag{2}$$

We want to solve this problem by reducing it to the heat equation. Start by introducing an appropriate function v such that the PDE in (2) reduces to $v_t = \Delta v$. Then rewrite the problem as a problem on v and find the expression for v. Finally deduce the solution for u.

Question 2 (Laplace equation, 10pts)

The fundamental solution for the two dimensional Laplace's equation is given by

$$\Phi(\boldsymbol{x}) = -\frac{1}{2\pi}\log|\boldsymbol{x}|, \quad |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2}$$

1. [5pts] *Provide the Green function for Laplace's equation, for the (hatched) region $\{(x_1, x_2) \mid x_1 > 0, x_2 > 0\}$ shown in Fig 2*

2. [5pts] *We consider the "bump" function $f(\boldsymbol{x}) \in C_c^2(\mathbb{R}^2)$ (i.e. smooth functions with compact support) defined as*

$$f(\boldsymbol{x}) = \begin{cases} \exp\left(-\frac{1}{1-|\boldsymbol{x}|^2}\right) & |\boldsymbol{x}|^2 < 1 \\ 0 & \textit{otherwise} \end{cases}$$

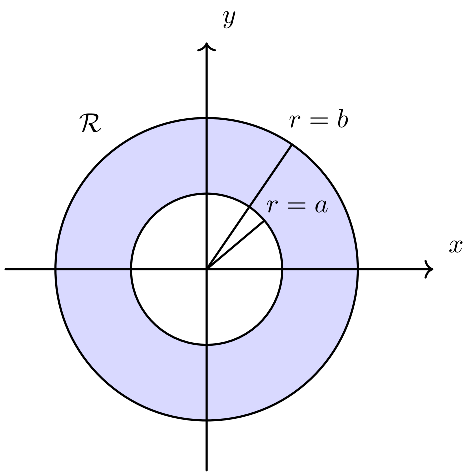

Figure 1: Annular region $\mathcal{R}$ used in Question 1

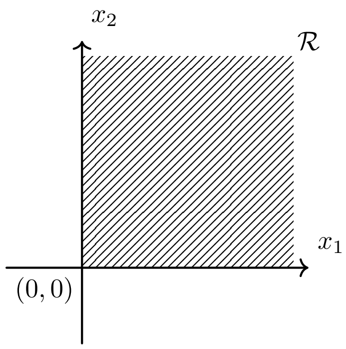

Figure 2: Region $\mathcal{R} = \{(x_1, x_2) \mid x_1 > 0, x_2 > 0\}$ used in Question 2

Provide the unique solution (vanishing at infinity) of Poisson's equation

$$\Delta u = -f(\boldsymbol{x}) \tag{3}$$

Question 3 (Theory, 10pts)

1. [5pts] *Using the energy method, show that the solution of problem* (2) *is unique.*
2. [5pts] *A function is called radial if its value at* $\boldsymbol{x}$ *depends only on* $|\boldsymbol{x}| = \sqrt{x_1^2 + \ldots + x_n^2}$. *Prove that a radial harmonic function on* $B = \{\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x}| < 1\}$ *is constant.*