Partial Differential Equations - MATH-UA 9263
Final III

Augustin Cosse

May 2022

Total: 39 points
Total time: 2h00
General instructions: The exam consists of 4 questions (each question consisting itself of 2 or 3 subquestions). Once you are done, make sure to write your name on each page, then take a picture of all your answers and send the file by email to acosse@nyu.edu. In case you have any question, you can ask those through the chat. Answer as many questions as you can starting with those you feel more confident with.

Question 1 (15pts)

We consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, & 0 < x < L, t > 0 \\ u(x,0) = 0, & 0 \leq x \leq L \\ u(0,t) = \sin(t) & t > 0 \\ u(L,t) = 0 & t > 0 \end{cases} \tag{1}$$

To solve this problem, we consider the function $v(x,t) = u(x,t) - \left(1 - \frac{x}{L}\right)\sin t$

1. [2pts] *Verify that the function* v *satisfies the new problem*

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial^2 v}{\partial x^2}(x,t) = f(x)g(t) & 0 < x < L, t > 0 \\ v(x,0) = v_0(x) & 0 \leq x \leq L \\ v(0,t) = 0 & t > 0 \\ v(L,t) = 0 & t > 0 \end{cases} \tag{2}$$

   and give the expressions of v_0, f *and* g.

2. [3pts] *We will solve (2) by looking for a solution of the form*

$$v(x,t) = \sum_{k=1}^{+\infty} T_k(t)\sin(\frac{k\pi x}{L}) \tag{3}$$

   By deriving the expression of the Fourier series for the periodic extension of the function $1 - \frac{x}{L}$, $0 \leq x \leq L$, *find the coefficients* c_k, $k \geq 1$ *such that*

$$1 - \frac{x}{L} = \sum_{k=1}^{+\infty} c_k \sin(\frac{k\pi x}{L}), \quad \textit{for } 0 < x < L$$

3. [3pts] *Using the previous question, show that the function* v *defined by (3) is a solution to (2) if for all* $k \geq 1$, *the function* $T_k(t)$ *is a solution of the ODE*

$$T_k'(t) + \left(\frac{k\pi}{L}\right)^2 T_k(t) = \gamma_k \cos t \tag{4}$$

   for appropriate γ_k.

4. [4pts] *Find the general solution of the ODE (4). Recall that the solution of a general linear ODE of the form (4) can be found from the solution of the homogeneous equation by using the method of variation of constants. I.e if the solution of the homogeneous equation has the form $T_k(t) = Af(t)$ then we substitute $v(t) = A(t)f(t)$ into (4) and find the expression for $A(t)$ that satisfies the equation.*

5. [3pts] *Using the initial condition in (2), find the complete expression of $T_k(t)$ and hence recover the final solution $u(x,t)$ from (1).*

Question 2 (10pts)

1. [5pts] *We consider Burger's equation*

$$\begin{cases} u_t + uu_x = 0 \qquad t \geq 0 \\ u(x,0) = \phi(x). \end{cases} \tag{5}$$

with the initial condition

$$\phi(x) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases}$$

Give the solution $u(x,t)$ of this system, including possible shock and rarefaction waves and sketch the characteristics (including fan-like characteristics if there are some) and shock wave(s). If there is a shock, detail the application of the Rankine Hugoniot condition.

2. [5pts] *Find the solution to the following conservation system*

$$\begin{cases} u_t + 3uu_x = 0 \\ u(x,0) = \phi(x) \end{cases} \tag{6}$$

where

$$\phi(x) = \begin{cases} 1 & x < 0 \\ 2 & 0 < x < 1 \\ 4 & x > 1 \end{cases}$$

Question 3 (10pts)

1. [6pts] *Question 3.1.*

   a) *State the (weak) maximum principle for a classical solution u of the partial differential equation $u_t = u_{xx}$ in the region $R = \{(x,t) \, : \, 0 < x < L, \; 0 < t \leq T\}$.*

   b) *Let $u(x,t)$ be a solution of the problem*

$$\begin{aligned} u_t - u_{xx} &= 0 & 0 < x < \pi/2, \quad 0 < t \leq T, \\ u(0,t) = e^{-t}, \quad u(\pi/2,t) &= 0 & 0 \leq t \leq T, \\ u(x,0) &= 1 - \frac{2x}{\pi} & 0 \leq x \leq \pi/2 \end{aligned}$$

   Prove that

$$0 \leq u(x,t) \leq e^{-t}\cos x \qquad \text{for } 0 < x < \pi/2, 0 < t \leq T$$

2. [2pts] *Give the definition of a Harmonic function*

3. [2pts] *Give the definition of a weak solution of the conservation law.*

Question 4 (4pts) *Give the general solution of the following Cauchy problem*

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = e^x, u_t(x,0) = \sin x \end{cases} \tag{7}$$