Partial Differential Equations - MATH-UA 9263
Final II

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Total: 28 points
Total time: 1h30
General instructions: The exam consists of 3 questions (each question consisting itself of 2 or 3 subquestions). Once you are done, make sure to write your name on each page, then take a picture of all your answers and send the file by email to acosse@nyu.edu. In case you have any question, you can ask those through the chat. Answer as many questions as you can starting with those you feel more confident with.

Question 1 (10pts)

1. [6pts] *We want to solve the following problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + cu = 0 & t > 0, 0 \leq x \leq L \\ \dfrac{\partial u}{\partial x}(t,0) = 0 & t > 0 \\ u(t,L) = 0 & t > 0 \\ u(0,x) = f(x) & 0 \leq x \leq L \end{cases} \tag{1}$$

where $c \in \mathbb{R}$ and $L > 0$ are fixed constants and $f(x)$ is piecewise C^1 on $[0, L]$.

   (a) *Find $\mu \in \mathbb{R}$ such that if we apply the change of variable $v(x,t) = e^{\mu t} u(x,t)$, we can reduce (1) to the heat equation*

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \tag{2}$$

   Write down the corresponding system (including initial and boundary conditions) on v

   (b) *We want to find the solutions of (2) of the form $v(x,t) = T(t)X(x)$. Find the general form of $T(t)$ and $X(x)$.*

   (c) *Show that if we look for $T(t)$ and $X(x)$ that satisfy the boundary conditions, we get an infinite number of solutions whose expressions are of the form $v_k(x,t) = T_k(t)X_k(x)$ where $k = 0, 1, 2, \ldots$. Give the expression of the the functions $T_k(t)$ and $X_k(x)$.*

   (d) *Find the solution v of the form*

$$v(x,t) = \sum_{k=1}^{\infty} T_k(t) X_k(x) \tag{3}$$

   Then derive the solution u of (1).

   (e) *We now assume $f(x) = \cos(\frac{\pi x}{2L})$, find the expression for $u(x,t)$.*

2. [4pts] *Solve the following general Cauchy problem (Hint: factor the operator as we did to derive d'Alembert's formula)*

$$\begin{cases} u_{xx} - 3u_{xt} - 4u_{tt} = 0 \\ u(x,0) = x^2, u_t(x,0) = e^x \end{cases} \tag{4}$$

Question 2 (10pts)

1. [5pts] *Solve the following initial value problem (Hint: consider $x+y$ and $x-y$)*

$$\begin{cases} y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0 \\ u(x,0) = f(x) \end{cases}$$

2. [5pts] *We consider the Riemann problem*

$$u_t + f(u)_x = 0 \quad \text{in } [0,\infty)\times\mathbb{R}, \quad u(0,x) = u_0(x)$$

with the flux function $f(u) = u^2(1-u)^2$. *Further let*

$$u_0(x) = \begin{cases} 2 & \text{if } x < 0, \\ 0 & \text{if } x > 0 \end{cases}$$

Construct a weak solution which satisfies the Rankine-Hugoniot condition.

Question 3 (10pts)

1. *Classify each of the following PDEs as hyperbolic, parabolic or elliptic [3pts].*
   (a) $u_{xx} - 5u_{xy} = 0$
   (b) $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$
   (c) $4u_{xx} + 6u_{xy} + 9u_{yy} = 0$
2. *Consider* $Q = \{(x,t) \mid 0 < x < L,\ t > 0\}$ *and* $\overline{Q}$ *the closure of* Q. *Assume that* u *and* v *are in* $C(\overline{Q}) \cap C^2(Q)$ *and respectively solve* $\partial_t u - \partial_x^2 u = f(x,t)$ *and* $\partial_t v - \partial_x^2 v = g(x,t)$ *on* Q. *Furthermore, assume that* $f < g$ *on* Q *and* $u \leq v$ *for* $t = 0$, *for* $x = 0$ *and for* $x = L$. *Show that* $u \leq v$ *on* Q *[Hint: use the proof of the weak maximum principle] [5pts]*
3. *Define the notion of harmonic function [2pts]*