Partial Differential Equations - MATH-UA 9263
Final I

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Total: 28 points
Total time: 1h15
General instructions: The exam consists of 3 questions (each question consisting itself of 2 or 3 subquestions). Once you are done, make sure to write your name on each page, then take a picture of all your answers and send the file by email to acosse@nyu.edu. In case you have any question, you can ask those through the chat. Answer as many questions as you can starting with those you feel more confident with.

Question 1 (10pts)

1. *Provide a full solution to the following Cauchy-Dirichlet problem for the heat equation [7pts]*

$$\begin{cases} u_t - u_{xx} = 0, & t > 0, 0 < x < L \\ u(x,0) = 2x + 1 & 0 \leq x \leq L \\ u(0,t) = 1 & t > 0 \\ u(L,t) = 3 & t > 0 \end{cases}$$

2. *Using polar coordinates, provide a solution to the following system*

$$\begin{cases} u_{xx} + u_{yy} = x^2 + y^2 & \textit{when } x^2 + y^2 < 1 \\ u(x,y) = 0, \quad \textit{when } x^2 + y^2 = 1 \end{cases}$$

(You want to look for solutions of the form $u(x,y) = x^2 + y^2$)[3pts]

Question 2 (10pts)

1. *We consider the first order problem*

$$\begin{cases} u_t + 2uu_x = 0 & t \geq 0 \\ u(x,0) = \phi(x) \end{cases} \tag{1}$$

with the initial condition

$$\phi(x) = \begin{cases} 3 & x < 0 \\ 4 & x > 0 \end{cases}$$

Give the solution to problem (1) including possible shocks. Provide a graphical representation of the characteristics as well as the expression of the shock curve if there is one [4pts].

2. *Find the solution to the following equations [6pts]*

$$u_x + u_y + u = 1, \quad u(x, x + x^2) = \sin x, \quad x > 0$$

$$(x - y)u_x + (x + y)u_y = u, \quad u(x,0) = x^2$$

Question 3 (8pts)

1. *Let* $u \in C^\infty(\mathbb{R}^3)$ *be a harmonic function on* $\mathbb{R}^3$

$$\Delta u(x) = 0, \quad x \in \mathbb{R}^3$$

Assume that $|u(x)| \leq \sqrt{|x|}$ *holds for all* x*, where* $x = (x_1, x_2, x_3)$ *and* $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$*. Show that* $u(x) = 0$ *for all* $x \in \mathbb{R}^3$ *[3pts].*

2. *Classify the following PDE as elliptic, hyperbolic or parabolic [2pts]*

$$-\partial_t^2 u(t,x) + 4\partial_t \partial_x u(t,x) - \partial_x^2 u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}$$

3. *Provide the solution of the Cauchy problem [3pts]*

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = e^x, \quad u_t(x,0) = \sin x$$