Introduction to optimisation

Recitation 05

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Question 1 *Show that the set of optimal solutions of a linear program defined on a compact polytope always contains one of the extreme points*

Question 2 *Consider the feasible sets given in Fig. 1. For each set, draw the convex hull of the set. Motivate your drawing with a short proof.*

Question 3 *Let* $P = \left\{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 \leq 2x_2, 2x_1 \geq 3\right\}$ *and consider the set* $P \cap \mathbb{Z}^2$*. Assume that we find* $x^* = (1.5, 0.8)$ *as a solution of a given LP on* P*. Find a valid inequality for* $P \cap \mathbb{Z}^2$ *cutting off* x^*

Question 4 *Let* $P = \left\{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 \geq 2, 4x_2 \leq x_1\right\}$ *and consider the set* $P \cap \mathbb{Z}^2$*. Assume that we find* $x^* = (2, 0.5)$ *as a solution of a given LP on* P*. Find a valid inequality for* $P \cap \mathbb{Z}^2$ *cutting off* x^*

Question 5 *Solve the following problem using LP relaxation and Chvátal-Gomory cuts*

$$\begin{array}{ll} \max & 6x_1 + 6x_2 \\ s.t. & x_1 + 2x_2 \leq 5 \\ & 2x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z}^2 \end{array}$$

Question 6 *Solve the following IP with Gomory's fractional cutting plane algorithm and represent the cut inequalities together with the original constraints in the* (x_1, x_2) *space.*

$$\begin{array}{ll} \max & x_1 + 2x_2 \\ s.t. & x_1 - 2x_2 \geq -2 \\ & x_1 + x_2 \leq 3 \\ & x_1, x_2 \in \mathbb{Z}_+ \end{array}$$

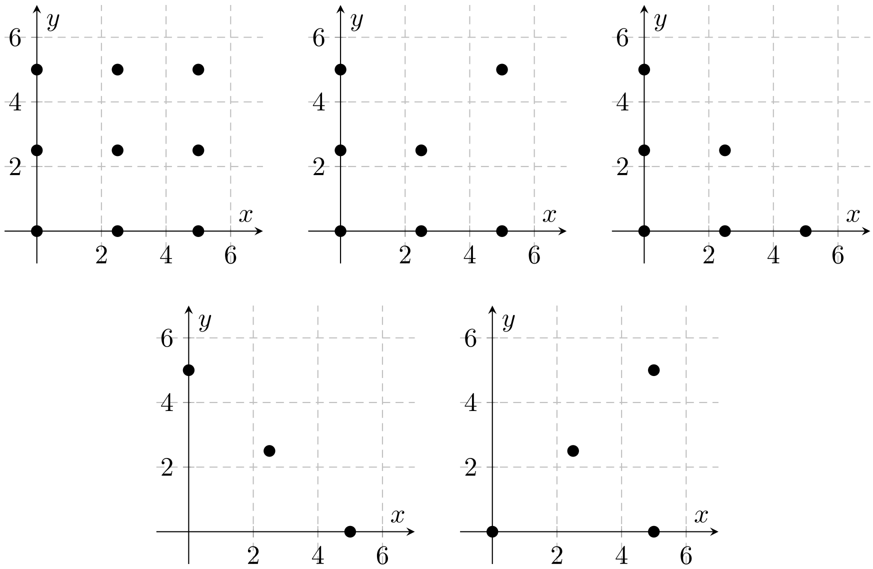

Figure 1: Feasible sets used in Question 2.

Question 7 *Consider the following integer linear program*

$$\begin{aligned} \max \quad & 4x_0 - 2x_1 \\ s.t. \quad & x_0 + 2x_1 \leq 2 \\ & 3x_1 \leq 2 \\ & 3x_0 - 3x_1 \leq 2 \\ & x_0, x_1 \in \mathbb{Z}_+ \end{aligned}$$

In the solution of the linear relaxation of the problem, the variables x_0, x_1 and the slack variable associated to the second constraint are in the basis.

1. *Calculate the optimal tableau using the* SIMPLEX *method*
2. *Find a Chvátal-Gomory cutting plane*
3. *Show that with this cutting plane, the optimal solution of the LP relaxation becomes infeasible.*

Question 8 *Prove that $y_2 + y_3 + 2y_4 \leq 6$ is valid for*

$$X = \left\{ y \in \mathbb{Z}_+^4 \mid 4y_1 + 5y_2 + 9y_3 + 12y_4 \leq 34 \right\}$$

Question 9 *Consider the problem*

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ s.t. \quad & x_1 + x_2 \geq 4 \\ & \frac{1}{2}x_1 + \frac{5}{2}x_2 \geq \frac{5}{2} \\ & x \in \mathbb{Z}_+^2 \end{aligned}$$

Show that $x^* = \left(\frac{15}{4}, \frac{1}{4}\right)$ is the optimal linear programming solution and find an inequality cutting off x^*

Question 10 *Solve*

$$\begin{aligned} \min \quad & 5x_1 + 9x_2 + 23x_3 - 4s \\ s.t. \quad & 20x_1 + 35x_2 + 95x_3 \geq 319 \\ & x \in \mathbb{Z}^3_+ \end{aligned}$$

using Chvátal-Gomory inequalities or Gomory's cutting plane algorithm.

Question 11 *Solve the following integer programs using Branch and Bound*

$$\begin{aligned} \max \quad & x_1 + 4x_2 & \\ s.t. \quad & x_1 + x_2 & \leq 7 \\ & -x_1 + 3x_2 & \leq 3 \\ & x_1, x_2 \in \mathbb{Z}_+ & \end{aligned} \qquad \begin{aligned} \max \quad & 6x_1 + 8x_2 & \\ s.t. \quad & 2x_1 + 4x_2 & \leq 36 \\ & 3x_1 - 4x_2 & \leq 40 \\ & x_1, x_2 \in \mathbb{Z}_+ & \end{aligned}$$

Question 12 *Solve the following integer programs using Branch and Bound (if there are several choices, always pick the variable with the smallest subindex as your branching variable)*

$$\begin{aligned} \max \quad & 2x_1 + x_2 + 3x_3 & \\ s.t. \quad & x_1 + x_2 + 3x_3 & \leq 17 \\ & 3x_1 + 2x_2 + 2x_3 & \leq 11 \\ & x_1, x_2, x_3 \in \mathbb{Z}_+ & \end{aligned} \qquad \begin{aligned} \max \quad & x_1 + x_2 + x_3 & \\ s.t. \quad & 5x_1 + 2x_2 + 3x_3 & \leq 42 \\ & 2x_1 + 7x_2 + 5x_3 & \leq 52 \\ & x_1, x_2, x_3 \in \mathbb{Z}_+ & \end{aligned}$$

Question 13 *Consider the enumeration tree shown in Fig . Assuming that the tree was generated for a minimization problem and that the upper and lower bounds obtained for each subproblem are indicated on the left and right side of the corresponding node:*

(i) Give the best possible upper and lower bounds on the optimal value

(ii) Which node could be pruned and which must be further explored

Question 14 *Consider the following integer program*

$$\begin{aligned} \max \quad & 9x_1 + 5x_2 & \\ s.t. \quad & 4x_1 + 9x_2 & \leq 35 \\ & x_1 & \leq 6 \\ & x_1 - 3x_2 & \geq 1 \\ & 3x_1 + 2x_2 & \leq 19 \\ & x_1, x_2 \in \mathbb{Z}_+ & \end{aligned}$$

Solve the problem using the Branch-and-Bound and represent the corresponding tree.

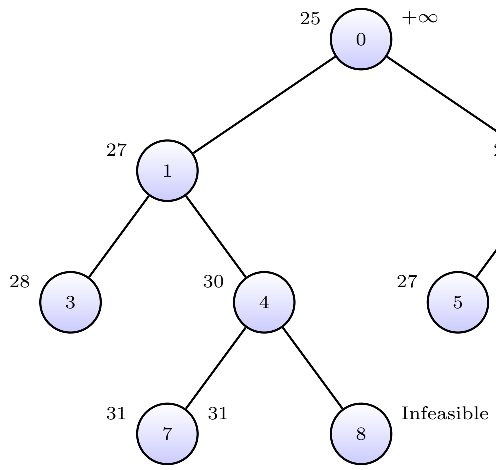

Figure 2: Enumeration tree used in Question 13.

References

[1] Pablo Pedregal, *Introduction to optimization*. Vol. 46. New York: Springer, 2004.

[2] Roy H Kwon, *Introduction to linear optimization and extensions with MATLAB*, CRC Press 2013.