Numerical Analysis
Lecture 3

Augustin Cosse

January 2022

This note was written as part of the series of lectures on Numerical Analysis delivered at ULCO in 2022-2023. The version is temporary. Please direct any comments or questions to augustin.cosse@univ-littoral.fr.

Introduction

Although the Laplace transform first official appearance can be traced back to the work of Bateman (1910) on radioactive decay [2], it is through the work of Doetsch that the transform can be considered to have really been introduced to the public.

Bateman considers the system of equations originally introduced by Rutherford, governing the amount of a primary substance P and the different products in a given quantity of radioactive matter.

$$\begin{cases} \frac{dP}{dt} &= -\lambda_1 P \\ \frac{dQ}{dt} &= \lambda_1 P - \lambda_2 Q \\ \frac{dR}{dt} &= \lambda_2 Q - \lambda_3 R \\ & \vdots \end{cases} \tag{1}$$

where $P(t), Q(t), R(t), S(t), \ldots$ encode the numbers of atoms of each substance. Although the original system of Rutherford can be solved explicitly (by means of the derivative of the logarithm), the direct solution quickly becomes laborious. As an alternative, Bateman suggests to introduce the quantities

$$p(x) = \int_0^\infty e^{-xt} P(t)\, dt, \quad q(x) = \int_0^\infty e^{-xt} Q(t)\, dt, \quad \ldots \tag{2}$$

Using integration by parts, he then notes that we can derive the relation

$$\begin{aligned}\int_0^\infty e^{-xt}\frac{dP}{dt}\,dt &= -P(0) + x\int_0^\infty e^{-xt}P(t)\,dt\\ &= -P(0) + xp(x)\end{aligned}$$

Substituting this in (1) gives the system

$$\begin{aligned}xp(x) - P_0 &= -\lambda_1 p\\ xq(x) - Q_0 &= \lambda_1 p - \lambda_2 q\\ xr(x) - R_0 &= \lambda_2 q(x) - \lambda_3 r(x)\\ &\vdots\end{aligned}$$

And if we assume that there is only one substance present initially ($Q_0 = R_0 = \ldots = 0$), we can then obtain the solutions for $p, q, r, \ldots$ at once

$$p(x) = \frac{P_0}{x+\lambda_1},\quad q(x) = \frac{\lambda_1 P_0}{(x+\lambda_1)(x+\lambda_2)},\quad r(x) = \frac{\lambda_1\lambda_2 P_0}{(x+\lambda_1)(x+\lambda_2)(x+\lambda_3)},\ldots$$

As was stated earlier, Bateman's use of the Laplace transform was essentially motivated by a desire to simplify Rutherford's approach and not as a tool leading to the discovery of new solutions and/or opening new research directions. A more serious discussion on the use of the Laplace transform in mathematics, and of its application to physics and engineering can be found in the work of Doetsch (see [4]). In his monograph, Doetsch introduces the Laplace transform by comparing it to the Fourier transform. Doetsch's comparison starts by focusing on the particular case of functions for which the classical Fourier transform is not defined (I discuss the case of those functions in the lecture on Fourier transforms). Examples of such functions include $f(t) = 1$ as well as $f(t) = e^{i\omega t}$. Doetsch then points out that those functions can be treated by two slight modifications of the Fourier transform:

i) The restriction to the halfspace $[0, +\infty]$ (which remains of practical interest, as in most applications, be it in physics and/or chemistry, we are interested in processes that begin at a specific instant)

ii) The applications of the transform to the family of functions $e^{-\gamma t}f(t)$ rather than to $f(t)$ only.

Restricting the transform to the half line $[0, +\infty]$ is needed for an integral of the functions $e^{-\gamma t}f(t)$ to be defined, and introducing the extension of the transform to a family of functions $\{e^{-\gamma t}f(t)\}_{\gamma\in\mathbb{R}}$ makes it possible to study the existence of the transform across the family (i.e. as a function of the parametrization γ) instead of simply deriving a yes/no conclusion on a single function.

With those two restrictions, the Fourier transform becomes

$$\int_0^\infty e^{-i\omega t}\left[e^{-\gamma t}f(t)\right]\,dt = \int_0^\infty e^{-st}f(t)\,dt = F(s),\quad s = \gamma + i\omega \tag{3}$$

One can check that the transform (3) can now be applied to the functions $f(t) = 1$ and $f(t) = e^{i\omega t}$ (under appropriate restrictions on the real part of s).

Because of its interesting properties, in particular when combined with differential operators, the Laplace transform reveals peculiarly useful in the analysis of differential equations. An example of such applications will be given at the end of the lecture.

Definition

Definition 1. *Suppose that f is a real or complex valued function of the (time) variable $t > 0$ and s is a real or complex parameter. We define the Laplace transform of f as*

$$\begin{aligned} F(s) = \mathcal{L}(f(t)) &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt \end{aligned}$$

whenever the limit exists.

Example 1. *if $f(t) = 1$ for all $t \geq 0$, then*

$$\begin{aligned} \mathcal{L}(f(t)) &= \int_0^\infty e^{-st}\,1\,dt \\ &= \lim_{\tau\to\infty}\left|\frac{e^{-st}}{-s}\right|_0^\tau \\ &= \left(\lim_{\tau\to\infty}\frac{e^{-s\tau}}{-s} + \frac{1}{s}\right) \\ &= \frac{1}{s} \end{aligned}$$

Provided that $\mathbb{R}\mathrm{e}\{s\} > 0$

Example 2.

$$\begin{aligned} \mathcal{L}(e^{i\omega t}) &= \int_0^\infty e^{-st}e^{i\omega t}\,dt \\ &= \lim_{\tau\to\infty}\left.\frac{e^{(i\omega-s)t}}{i\omega - s}\right|_0^\tau \\ &= \frac{1}{s - i\omega} \end{aligned}$$

Provided that $\mathbb{R}\mathrm{e}\{s\} > 0$

Note that using $\frac{e^{i\omega t}+e^{-i\omega t}}{2} = \cos\omega t$ and using the linearity of the Laplace transform, Example 2 can be used to derive the following additional transforms

$$\begin{aligned}
\mathcal{L}(\cos \omega t) = \mathcal{L}\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) &= \frac{\mathcal{L}\left(e^{i\omega t}\right) + \mathcal{L}\left(e^{-i\omega t}\right)}{2} \\
&= \frac{1}{2}\left(\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\right) \\
&= \frac{s}{s^2 + \omega^2}
\end{aligned}$$

$$\mathcal{L}(\sin \omega t) = \frac{1}{2i}\left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right) = \frac{\omega}{s^2 + \omega^2}.$$

Although the Laplace transform can be applied to many functions, there are functions for which the integral does not converge

Example 3. *For the function* $f(t) = e^{t^2}$

$$\lim_{\tau \to \infty} \int_0^\tau e^{-st} e^{t^2}\, dt = \lim_{\tau \to \infty} \int_0^\tau e^{t^2 - st}\, dt = \infty$$

To guarantee the existence of the Laplace transform, we will usually require two properties:

- Piecewise continuity
- Exponential Order

Defining the notion of piecewise continuity requires to define the notion of jump discontinuity

Definition 2. *A function f has a jump discontinuity at a point t_0 if both the limits*

$$\lim_{t \to t_0^-} f(t) = f(t_0^-), \quad \text{and} \quad \lim_{t \to t_0^+} f(t) = f(t_0^+)$$

exist (as finite numbers) and $f(t_0^-) \neq f(t_0^+)$. Here $t \to t_0^-$ and $t \to t_0^+$ mean that $t \to t_0$ from the left and from the right, respectively.

Definition 3. *A function f is piecewise continuous on the interval $[0, +\infty)$ if (i) $\lim_{t \to 0^+} f(t) = f(0^+)$ exists and (ii) f is continuous on every finite interval $(0, b)$ except possibly at a finite number of points $\tau_1, \tau_2, \ldots, \tau_n$ in $(0, b)$ at which f has a jump discontinuity.*

Another important condition for the existence of the Laplace transform is the fact that the function should not grow too fast. I.e., for the integral to remain finite, the function growth should not "dominate" the decrease of the e^{-st}

Definition 4. *A function f has exponential order α if there exist constants $M > 0$ and α such that for some $t_0 \geq 0$*

$$|f(t)| \leq Me^{\alpha t}, \quad t \geq t_0$$

From the two conditions 3 and 4, we can now state our main existence theorem for the Laplace transform

Theorem 1. *If f is piecewise continuous on $[0, +\infty)$ and of exponential order α, then the Laplace transform $\mathcal{L}(f)$ exists for $\operatorname{Re}\{s\} > \alpha$ and converges absolutely. We define the class L as the set of functions defined on $(0, +\infty)$ for which the Laplace transform exists for some value of s.*

We now discuss some of the important properties of the Laplace transform

- Linearity. If $f_1 \in L$ for $\operatorname{Re}(s) > \alpha$, $f_2 \in L$ for $\operatorname{Re}(s) > \beta$ then $f_1 + f_2 \in L$ for $\operatorname{Re}(s) > \max\{a, b\}$ and

$$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2)$$

  for arbitrary constants c_1, c_2.

- Scaling. For any real $a > 0$, we have

$$\begin{aligned}\mathcal{L}(f(at))(s) &= \int_0^\infty \frac{1}{a} af(at)e^{-st}\, dt = \int_0^\infty \frac{1}{a} f(\gamma)e^{-\frac{s\gamma}{a}}\, d\gamma \\ &= \frac{1}{a}\mathcal{L}(f)\left(\frac{s}{a}\right) = \frac{1}{a}F\left(\frac{s}{a}\right)\end{aligned}$$

- Shift. For any $b \in \mathbb{R}$, we have

$$\mathcal{L}\left(e^{-bt}f(t)\right) = \int_0^{+\infty} e^{-st}e^{-bt}f(t)\, dt \tag{4}$$

$$= \int_0^\infty e^{-(s+b)t}\, f(t)\, dt \tag{5}$$

$$= F(s+b) \tag{6}$$

Most of the Laplace transforms can be derived from a few elementary transforms

Example 4. *As an example, we consider the function $f(t) = t$. We have*

$$\begin{aligned}\int_0^T te^{-st}\,dt &= \left[-\frac{t}{s}e^{-st}\right]_0^T - \int_0^T -\frac{1}{s}e^{-st}\,dt \\ &= -\frac{T}{s}e^{-sT} - \left[\frac{1}{s^2}e^{-st}\right]_0^T \\ &= -\frac{T}{s}e^{-sT} - \frac{1}{s^2}e^{-sT} + \frac{1}{s^2}\end{aligned}$$

Taking the limit $T \to \infty$ we recover $F(s) = 1/s^2$

The previous example can be used to find the Laplace transform for the function $f(t) = t^n$ as well as the transform of the function $f(t) = \sum_{k=0}^{n} a_k t^k$.

Example 5.

$$\begin{aligned}\mathcal{L}(t^n) &= \int_0^\infty t^n e^{-st}\,dt \\ &= \left[-\frac{t^n}{s}e^{-st}\right]_0^\infty + \int_0^\infty \frac{nt^{n-1}}{s}e^{-st}\,dt \\ &= \frac{n}{s}\mathcal{L}(t^{n-1})\end{aligned}$$

Substituting the value $n = 2$ in the recurrence above, we get

$$\mathcal{L}\left(t^2\right) = \frac{2}{s}\mathcal{L}(t) = \frac{2}{s^3}$$

And for any n,

$$\mathcal{L}(t^{n+1}) = \frac{n!}{s^{n+1}} \tag{7}$$

Applying the Laplace transform to infinite series should be done with caution as it requires a good understanding of the intrication between the integral and the series. Considering an infinite series

$$f(t) = \sum_{n=0}^{\infty} a_n t^n$$

the following proposition holds

Proposition 1. *If the series above converges for $t \geq 0$ with $|a_n| \leq \frac{K\alpha^n}{n!}$ for all n sufficiently large and $\alpha > 0$, $K > 0$, then*

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} a_n \mathcal{L}\{t^n\} = \sum_{n=0}^{\infty} \frac{a_n n!}{s^{n+1}} \quad (\mathrm{Re}\{s\} > \alpha)$$

Proof. Although we cannot guarantee that the integral and the sum can be swapped, we can however try to show that the difference

$$\left|\mathcal{L}(f) - \sum_{n=0}^{N} a_n \mathcal{L}(t^n)\right|$$

goes to zero as $N \to \infty$. I.e. that for all $\epsilon > 0$, there exists $\underline{\mathrm{N}} \in \mathbb{N}$ such that for all $N \geq \underline{\mathrm{N}}$

$$\left|\mathcal{L}(f) - \sum_{n=0}^{N} a_n \mathcal{L}(t^n)\right| < \varepsilon$$

For any finite N, we have

$$\begin{aligned}\left|\mathcal{L}\left(f - \sum_{n=0}^{N} a_n t^n\right)\right| &\leq \int_0^\infty \left|f - \sum_{n=0}^{N} a_n t^n\right| \left|e^{-st}\right| \, dt \\ &\leq \int_0^\infty \left|f - \sum_{n=0}^{N} a_n t^n\right| e^{-\operatorname{Re}\{s\}t} \, dt\end{aligned}$$

Using the bound on the coefficients a_n, we get, for any non negative t,

$$\begin{aligned}\left|f - \sum_{n=0}^{N} a_n t^n\right| &= \left|\sum_{n=N+1}^{\infty} a_n t^n\right| \\ &\leq \sum_{n=N+1}^{\infty} \frac{K\alpha^n}{n!} t^n \\ &\leq K\left(e^{\alpha t} - \sum_{n=0}^{N} \frac{\alpha^n t^n}{n!}\right)\end{aligned}$$

Substituting this in the integral, and integrating by parts n times, we have

$$\begin{aligned}\left|\mathcal{L}\left(f - \sum_{n=0}^{N} a_n t^n\right)\right| &\leq K\frac{1}{\operatorname{Re}\{s\} - \alpha} - K\sum_{n=0}^{N} \frac{\alpha^n}{\operatorname{Re}\{s\}^{n+1}} \\ &\leq K\frac{1}{\operatorname{Re}\{s\} - \alpha} - \frac{K}{\operatorname{Re}\{s\}} \sum_{n=0}^{N} \frac{\alpha^n}{\operatorname{Re}\{s\}^n}\end{aligned}$$

To bound the last line, we use the geometric sum

$$\begin{aligned}\frac{K}{\operatorname{Re}\{s\}} \sum_{n=0}^{N} \frac{\alpha^n}{\operatorname{Re}\{s\}^n} &= \frac{K}{\operatorname{Re}\{s\}} \frac{1 - \left(\frac{\alpha}{\operatorname{Re}\{s\}}\right)^N}{1 - \left(\frac{\alpha}{\operatorname{Re}\{s\}}\right)} \\ &= \frac{K}{\operatorname{Re}\{s\}} \frac{1 - \left(\frac{\alpha}{\operatorname{Re}\{s\}}\right)^N}{\operatorname{Re}\{s\} - \alpha} \operatorname{Re}\{s\} \\ &= K\frac{1 - \left(\frac{\alpha}{\operatorname{Re}\{s\}}\right)^N}{\operatorname{Re}\{s\} - \alpha}\end{aligned}$$

Taking the limit $N \to \infty$ and using the fact that $\mathrm{Re}\{s\} > \alpha$, we get

$$\lim_{N\to\infty} \left| \mathcal{L}\left(f - \sum_{n=0}^{N} a_n t^n \right) \right| \leq 0 \tag{8}$$

From this, since f encodes the whole infinite sum, we can write

$$\begin{aligned}
\mathcal{L}(f) &= \sum_{n=0}^{N} a_n \mathcal{L}\{t^n\} + \left(\mathcal{L}(f) - \sum_{n=0}^{N} a_n \mathcal{L}\{t^n\} \right) \\
&= \sum_{n=0}^{N} a_n \mathcal{L}\{t^n\} + \mathcal{L}\left(f - \sum_{n=0}^{N} a_n t^n \right)
\end{aligned}$$

The conclusion follows from taking the limit $N \to \infty$ and using (8).

□

As indicated above, the Laplace transform plays an important role in the analysis of ordinary differential equations, in part because of its particularly simple application to differential operators.

Theorem 2.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt = -f(0) + sF(s)$$

The result follows from an integration by parts

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \left| f(t)e^{-st} \right|_0^\infty + \int_0^\infty s e^{-st} f(t)\, dt \\
&= -f(0) + sF(s)
\end{aligned}$$

Similarly, if $f(t)$ is twice differentiable, then

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sF(0) - F'(0)$$

again, the result can be established by integrating by parts twice

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= \int_0^\infty e^{-st} f''(t)\, dt \\
&= \left| f'(t)e^{-st} \right|_0^\infty + \int_0^\infty s e^{-st} f'(t)\, dt \\
&= -f'(0) + \left| s e^{-st} f(t) \right|_0^\infty + \int_0^\infty s^2 e^{-st} f(t)\, dt \\
&= -f'(0) - sf(0) + s^2 F(s)
\end{aligned}$$

More generally, we get

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

One can derive a similar relation when considering the derivative of the Laplace transform.

Theorem 3. *If* $\mathcal{L}\{f(t)\} = F(s)$ *then* $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$ *and in general*

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Proof. Starting with the definition of the Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

and taking the derivative with respect to s, we get

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty -te^{-st} f(t)\, dt$$

where we assume absolute convergence to interchange the derivative and the improper integral. From this, we get $-F'(s) = \mathcal{L}\{tf(t)\}$

The higher order case follows directly as

$$\begin{aligned}
\frac{d^n F}{ds^n} &= \frac{d^n}{ds^n}\int_0^\infty f(t)e^{-st}\, dt \\
&= \int_0^\infty f(t)\frac{d^n}{ds^n} e^{-st}\, dt \\
&= (-1)^n \int_0^\infty t^n f(t)e^{-st}\, dt = (-1)^n \mathcal{L}\{t^n f(t)\}
\end{aligned}$$

□

When the Laplace transform is integrable, provided that the integrals

$$\int_\gamma^\infty \int_0^\infty f(t)e^{-st}\, dt\, ds, \quad \text{and} \quad \int_0^\infty \int_\gamma^\infty f(t)\, e^{-st}\, dt\, ds$$

are absolutely convergent, we can apply Fubini's theorem which gives

$$\begin{aligned}
\int_\gamma^\infty F(s)\, ds &= \int_0^\infty \int_\gamma^\infty f(t)e^{-st}\, ds\, dt \\
&= \int_0^\infty \left| \frac{e^{-st}}{-t} f(t) \right|_\gamma^\infty dt \\
&= \int_0^\infty \frac{e^{-\gamma t}}{t} f(t)\, dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}
\end{aligned}$$

The previous connection can be summarized by Proposition 2 below

Proposition 2. *Let $F(s) = \mathcal{L}\{f\}$ and let us assume that the integrals*

$$\int_\gamma^\infty \int_0^\infty f(t)e^{-st}\, dt\, ds, \quad \text{and} \quad \int_0^\infty \int_\gamma^\infty f(t)e^{-st}\, dt\, ds$$

are both absolutely convergent, then the following relation holds

$$\int_\gamma^\infty F(s)\, ds = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

Inverse transform

Most traditional operations have inverses. Addition has subtraction, multiplication has division. Differentiation has integration. Similarly we can define an inverse for the Laplace transform. The first important thing to notice in order to define the inverse is that the unilateral Laplace transform relies on integrating on $[0, +\infty]$. Defining an inverse thus only makes sense on the half line.

Consider the following example with $\sin \omega t$. Integrating by parts, we have

$$\begin{aligned}
\int_0^\infty \sin \omega t\, e^{-st}\, dt &= \int_0^\infty -\frac{i}{2}\left(e^{i\omega t} - e^{-i\omega t}\right) e^{-st}\, dt \\
&= \int_0^\infty -\frac{i}{2}\left(e^{t(i\omega - s)} - e^{t(-i\omega - s)}\right)\, dt \\
&= \left| -\frac{i}{2}\frac{1}{i\omega - s}e^{(i\omega - s)t} + \frac{i}{2}\frac{1}{-i\omega - s}e^{(-i\omega - s)t}\right|_0^\infty \\
&= -\frac{i}{2}\left(\frac{1}{-i\omega + s} - \frac{1}{s + i\omega}\right) \\
&= -\frac{i}{2}\left(\frac{2i\omega}{s^2 + \omega^2}\right) = \frac{\omega}{s^2 + \omega^2}
\end{aligned}$$

From the above, it is therefore tempting (and in this case the intuition is correct) to define the inverse Laplace transform of the function $\frac{\omega}{s^2+\omega^2}$ as

$$\mathcal{L}^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\} = \sin \omega t, \quad t \geq 0$$

As indicated by the additional example below however, additional care has to be taken though. Indeed, consider the function

$$g(t) = \begin{cases} \sin \omega t & t > 0 \\ 1 & t = 0 \end{cases}$$

Clearly we again have

$$\int_0^\infty g(t)e^{-st}\,dt = \int_0^\infty \sin(\omega t)e^{-st}\,dt$$
$$= \frac{\omega}{s^2+\omega^2}$$

From the two examples above, we see that ensuring uniqueness of the inverse Laplace transform will require additional care. Such care essentially reduces to the continuity of the function $f(t)$ (Technically, we will say that two functions which have identical Laplace transform (in a right halfspace) differ at most by a null function, that is to say a function $n(t)$ whose integral $\int_0^\tau n(t)\,dt = 0$ vanishes identically for all upper limit $\tau \geq 0$)

Definition 5. *if $f(t)$ has the Laplace transform $F(s)$ that is $\mathcal{L}(f(t)) = F(s)$ then the inverse Laplace transform of $F(s)$ can be defined by*

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \quad t \geq 0$$

Moreover, on the space of continuous functions defined on $[0,\infty)$, this inverse transform is unique.

Partial Fractions

It is often the case, when considering applications of the inverse Laplace transform, to encounter expressions that do not correspond to the transform of any elementary function (such as t, $\sin t$, $\cos t$, $\ldots$) or any simple modification of such functions but can nonetheless read as a sum of a few simple transforms. Consider for example the function

$$F(s) = \frac{1}{2(s-1)} + \frac{1}{2(s+1)}$$

In this case, it remains possible to derive the inverse transform of $F(s)$ by

1. Recalling that

$$\mathcal{L}\{1\} = \int_0^\infty 1\;e^{-st}\,dt = \frac{1}{s}$$

2. Applying the shift property (6)

$$\mathcal{L}\left\{e^{-bt}f(t)\right\} = F(s+b)$$

which, in the case $f(t) = 1$ considered above, gives

$$\int_0^\infty 1e^{-bt}e^{-st}\,dt = \frac{1}{s+b}$$

From those two steps, we can write

$$\mathcal{L}^{-1}\left(\frac{1}{2(s-1)}+\frac{1}{2(s+1)}\right)=\frac{1}{2}e^{t}+\frac{1}{2}e^{-t} \tag{9}$$

Applying the inverse transform to a sum of fractions of the form $\frac{1}{s+a}$ is thus easy.

In some cases, such as (9), the sum of simple transforms is explicit. In some others, such as discussed below, the inverse transforms are not directly visible. Consider for example the function $G(s)$ defined as

$$G(s)=\frac{s}{s^2+4s+1} \tag{10}$$

When facing a function of the general form

$$G(s)=\frac{P(s)}{Q(s)}$$

where the degree of $Q(s)$ is greater than the degree of $P(s)$, one can decompose the fraction into a sum of elementary terms of the form

$$G(s)=G_1(s)+G_2(s)+\dots \tag{11}$$

that are defined, from the factorization of $Q(s)$ as follows:

- For each linear factor of the form $as+b$ in $Q(s)$ there corresponds a term $\frac{A}{as+b}$ in the decomposition (11)
- For each repeated linear factor of the form $(as+b)^n$, there corresponds a sum of terms of the form

$$\frac{A_1}{(as+b)}+\frac{A_2}{(as+b)^2}+\dots+\frac{A_n}{(as+n)^n}$$

- For each quadratic factor of the form as^2+bs+c, there corresponds a term of the form

$$\frac{As+B}{as^2+bs+c}$$

- Finally for every repeated quadratic factor of the form $(as^2+bs+c)^n$, there corresponds a sum of terms of the form

$$\frac{A_1s+B_1}{as^2+bs+c}+\frac{A_2s+B_2}{(as^2+bs+c)^2}+\dots+\frac{A_ns+B_n}{(as^2+bs+c)^n}$$

The coefficients A_i, B_i have to be determined by identifying $G(s)$ and its expansion. Finally, from the expansion of $G(s)$, one can compute the inverse transform by relying on the direct transform of simple functions $(1, \sin \omega t, \cos \omega t, \ldots)$ and the shift property. For trigonometric functions, recall that we have

$$\mathcal{L}\left(e^{at} \cos \omega t\right) = \frac{s-a}{(s-a)^2 + \omega^2} \tag{12}$$

$$\mathcal{L}\left(e^{at} \sin \omega t\right) = \frac{\omega}{(s-a)^2 + \omega^2} \tag{13}$$

$$\mathcal{L}\left(e^{at} \cosh \omega t\right) = \frac{s-a}{(s-a)^2 - \omega^2} \tag{14}$$

$$\mathcal{L}(e^{at} \sinh \omega t) = \frac{\omega}{(s-a)^2 - \omega^2} \tag{15}$$

We now examine two particular examples of application of the inverse transform.

Example 6. *Consider the function*

$$F(s) = \frac{1}{s^2 + 2s + 5}$$

Using the decomposition

$$\begin{aligned} F(s) &= \frac{1}{2} \frac{2}{s^2 + 2s + 1 + 4} \\ &= \frac{1}{2} \frac{2}{(s+1)^2 + 4} \end{aligned}$$

and combining this with the transform (13), *we get*

$$\mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{2} e^{-t} \sin 2t$$

Example 7. *We now consider the function*

$$F(s) = \frac{1}{s^2 + 3s + 2}$$

In this case, we can either use the decomposition

$$F(s) = 2\frac{1/2}{(s + \frac{3}{2})^2 - \frac{1}{4}}$$

From which, applying (15), one can obtain

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= 2e^{-3t/2}\sinh\frac{t}{2}\\ &= 2e^{-3t/2}\frac{\left(e^{t/2} - e^{-t/2}\right)}{2}\\ &= e^{-t} - e^{-2t}\end{aligned}$$

An alternative is to use the partial fraction expansion

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

and requiring the equality

$$\frac{A(s+2) + B(s+1)}{(s+1)(s+2)} = \frac{1}{s^2 + 3s + 2} = F(s)$$

from which the coefficients A and B can be found by either matching the numerators from the left and right-hand sides,

$$\begin{aligned}A(s+2) + B(s+1) = 1 &\Rightarrow (A+B)s + (2A+B) = 0 \cdot s + 1\\ &\Rightarrow \begin{cases}(A+B) = 0\\ 2A + B = 1\end{cases}\end{aligned}$$

or by noting that, since the equation

$$A(s+2) + B(s+1) = 1$$

must be satisfied for every s, in particular we must have, for $s = -2$ and $s = -1$,

$$\begin{aligned}B(-2+1) = -B = 1\\ A(-1+2) = A = 1\end{aligned}$$

From either approaches, one can thus derive a decomposition for $F(s)$ given by

$$F(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

on which we can apply the inverse transform to finally recover

$$\mathcal{L}^{-1}\{F(s)\} = e^{-t} - e^{-2t}$$

Resolution of differential equations

We consider the simple R-L-C circuit shown in Fig. 1, consisting of (i) an inductance L, a resitance R and a capacitance C (each assumed to be constant). We are interested in the recovery of the current as a function of time, $I(t)$. Relying on Kirchoff's voltage law, which states that the voltage drops across the individual components must equal the voltage $E(t)$ applied to the system, we can derive the following equation for $I(t)$,

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(t')\,dt' = E(t) \tag{16}$$

Neglecting the capacitor and assuming a sinusoidal voltage, $E(t) = E_0 \sin\omega t$, we get

$$L\frac{dI}{dt} + RI = E_0 \sin\omega t \tag{17}$$

Applying the Laplace transform to the left and right-hand sides, we get

$$(Ls + R)\mathcal{L}\{I\} = \frac{E_0\omega}{s^2 + \omega^2}$$
$$\mathcal{L}(I) = \frac{E_0\omega}{(Ls + R)(s^2 + \omega^2)}$$

Now using a partial fraction expansion for $\mathcal{L}(I)$ and computing the inverse transform, we recover

$$\mathcal{L}(I) = \frac{A}{s + R/L} + \frac{Bs + C}{s^2 + \omega^2}$$

where

$$A = \frac{E_0 L\omega}{L^2\omega^2 + R^2}, \quad B = \frac{-E_0 L\omega}{L^2\omega^2 + R^2}, \quad C = \frac{E_0 R\omega}{L^2\omega^2 + R^2}$$

Taking the inverse transform finally gives

$$I(t) = \frac{E_0 L\omega}{L^2\omega^2 + R^2} e^{-Rt/L} + \frac{E_0 R}{L^2\omega^2 + R^2}\sin\omega t - \frac{E_0 L\omega}{L^2\omega^2 + R^2}\cos\omega t \tag{18}$$

References

[1] Joel L. Schiff, *The Laplace transform, Theory and Applications*, Undergraduate Texts in Mathematics, Springer Science+Business Media LLC, 1988.

[2] Harry Bateman, *The solution of a system of differential equations occurring in the theory of radio-active transformations*, Proc. Cambridge Phil. Soc., 1908, vol. 15, pages 423–427.

[3] Phil Dyke, *An Introduction to Laplace Transforms and Fourier Series*, Second Edition, Springer Undergraduate Mathematics Series, 1999.

[4] Gustav Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*. Springer Science & Business Media, 2012.

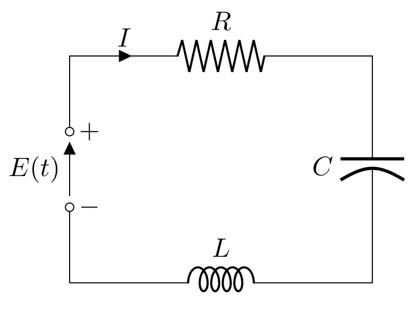

Figure 1: A simple R-L-C circuit.