Numerical Analysis
Lecture 2

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# Introduction

Polynomials have been used for years because of there beautiful properties and their ability to approximate functions. A classical problem which has a particularly long history in mathematics is the problem of learning the coefficients of a polynomial of degree $n$ (hence $n+1$ parameters) whose values are prescribed at $n+1$ distinct points. In this lecture, we will start by discussing the classical theory of polynomial interpolation including the resulting error. We will also cover the notion of approximation of a function by a polynomial. Finally we will briefly address the interpolation of trigonometric polynomials.

## General setting

The general idea of interpolation can be traced back to the seleucid period and the ephemerides found on ancient astronomical cuneiform tablets from the cities of Uruk and Babylon [4].

A first motivation for the use of polynomials as efficient objects in the approximation of functions can be found in Weierstrass approximation theorem,

**Theorem 1.** *[Weierstrass] Given $f : [a, b] \mapsto \mathbb{R}$ continuous and an arbitrary $\varepsilon > 0$, there exists an algebraic polynomial $p$ such that*

$$|f(x) - p(x)| \leq \varepsilon, \quad \forall x \in [a, b]$$

*Proof.* The proof unfolds relatively smoothly. The only technical step is an equality relating the variance of a binomial variable to its binomial expansion. Without loss of generality, we can assume $[a,b] = [0,1]$ (the general case can be obtained from the $[0,1]$ case via rescaling). We consider the sequence of polynomials $p_N(x), N \in \mathbb{N}$ defined as

$$p_N(x) = \sum_{n=0}^{N} f\left(\frac{n}{N}\right)\binom{N}{n}x^n(1-x)^{N-n} = \sum_{n=0}^{N} f\left(\frac{n}{N}\right)B_{n,N}(x)$$

obtained from the Bernstein polynomials

$$B_{n,N}(x) = \binom{N}{n}x^n(1-x)^{N-n}$$

To get a bound on $|f(x) - p_N(x)|$, we expand $f(x)$ as

$$\begin{aligned} f(x) &= f(x)(x + (1-x))^N \\ &= f(x)\sum_{n=0}^{N}\binom{N}{n}x^n(1-x)^{N-n} \end{aligned}$$

From the definition of the Bernstein polynomials, we can then write

$$f(x) - p_N = \sum_{n=0}^{N}\left(f(x) - f\left(\frac{n}{N}\right)\right)\binom{N}{n}x^n(1-x)^{N-n} \tag{1}$$

We will now show that for every $\varepsilon > 0$ the degree of $p_N$ can be chosen so that $|f(x) - p_N(x)| \leq \varepsilon$ for all $x \in [0,1]$. Recall that a function $f$ is continuous at $x_0 \in I$ if $\forall\varepsilon > 0$, $\exists\delta(x_0) > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$ for all $x \in I$ for which $|x - x_0| < \delta$. A function is uniformly continuous if for all $\varepsilon > 0$ $\exists$ $\delta$ such that

$$|f(x) - f(x_0)| < \varepsilon, \quad \text{for all } x, x_0 \in I \quad \text{s.t.} \quad |x - x_0| < \delta$$

In our case, the continuity of $f$ on the whole interval $[0,1]$ implies that for $\varepsilon > 0$, $\exists\delta > 0$, such that for all $x, y \in [0,1]$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$ (i.e continuity on a bounded closed interval implies uniform continuity on that same interval).

Given the uniform continuity, let $\varepsilon > 0$ and let $\delta$ be such that $|f(x) - f(y)| < \varepsilon$ as soon as $|x - y| < \delta$.

To control the deviation between $f(x)$ and $p_N(x)$, we divide the index set in the binomial expansion (1),

$$\begin{aligned} f(x) - p_N(x) = &\sum_{|x-\frac{n}{N}|<\delta}\left(f(x) - f\left(\frac{n}{N}\right)\right)\binom{N}{n}x^n(1-x)^{N-n} \\ &+ \sum_{|x-\frac{n}{N}|\geq\delta}\left(f(x) - f\left(\frac{n}{N}\right)\right)\binom{N}{n}x^n(1-x)^{N-n} \end{aligned}$$

Applying the triangle inequality to the first term and using the continuity of $f$, we get

$$\begin{aligned}
&\left| \sum_{|x-\frac{n}{N}|<\delta} \left(f(x) - f\left(\frac{n}{N}\right)\right) \binom{N}{n} x^n (1-x)^{N-n} \right| \\
&\le \sum_{|x-\frac{n}{N}|<\delta} \left|f(x) - f\left(\frac{n}{N}\right)\right| \binom{N}{n} x^n (1-x)^{N-n} \\
&\le \frac{\varepsilon}{2} \sum_{|x-\frac{n}{N}|<\delta} \binom{N}{n} x^n (1-x)^{N-n} \\
&\le \frac{\varepsilon}{2} \sum_{n=0}^{N} \binom{N}{n} x^n (1-x)^{N-n} \\
&= \frac{\varepsilon}{2} \left(x + (1-x)\right)^N = \frac{\varepsilon}{2}
\end{aligned}$$

For the second term, we use

$$\left| \sum_{|x-\frac{n}{N}|\ge\delta} \left(f(x) - f\left(\frac{n}{N}\right)\right) \binom{N}{n} x^n (1-x)^{N-n} \right| \tag{2}$$

$$\le \sum_{\frac{\left(x-\frac{n}{N}\right)^2}{\delta^2}\ge 1} \left|f(x) - f\left(\frac{n}{N}\right)\right| \binom{N}{n} x^n (1-x)^{N-n} \tag{3}$$

$$\le 2M \sum_{\frac{\left(x-\frac{n}{N}\right)^2}{\delta^2}\ge 1} \binom{N}{n} x^n (1-x)^{N-n} \tag{4}$$

Where $M = \sup_{x\in[a,b]} f(x)$. To get rid of the binomial factor, we use the following trick

$$2M \sum_{\frac{\left(x-\frac{n}{N}\right)^2}{\delta^2}\ge 1} \binom{N}{n} x^n (1-x)^{N-n} \le 2M \sum_{\frac{\left(x-\frac{n}{N}\right)^2}{\delta^2}\ge 1} \frac{\left(x-\frac{n}{N}\right)^2}{\delta^2} \binom{N}{n} x^n (1-x)^{N-n} \tag{5}$$

$$\le \frac{2M}{\delta^2} \sum_{n=0}^{N} \left(x - \frac{n}{N}\right)^2 \binom{N}{n} x^n (1-x)^{N-n} \tag{6}$$

$$\le \frac{2M}{\delta^2 N^2} \sum_{n=0}^{N} (n - Nx)^2 \binom{N}{n} x^n (1-x)^{N-n} \tag{7}$$

The second factor in the last line (highlighted in blue), is the variance of a binomial variable with parameter (number of trials) $N$ and probability of success $x$. This variance can equivalently read as $Nx(1-x)$. Substituting this in (7), we thus get

$$(7) \le \frac{2M}{\delta^2 N^2} Nx(1-x) = \frac{2M}{\delta^2 N} x(1-x) \le \frac{M}{\delta^2 N} \tag{8}$$

Combining (8) and (2), we get

$$|f(x) - p_N(x)| \leq \frac{\varepsilon}{2} + \frac{M}{2N\delta^2}$$

In particular, from this, we see that we can always choose $\delta$ and $N$ such that $|f(x) - p_N(x)| < \varepsilon$ for all $x \in [0,1]$.

□

## The interpolation problem

For $n \in \mathbb{N} \cap \{0\}$, we let $\mathbb{P}_n$ denote the linear space of polynomials

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

for a real (or complex) variable $x$ and with real (or complex) coefficients $a_0, a_1, \ldots a_n$. A polynomial $p \in \mathbb{P}_n$ is said to be of degree $n$ if $a_n \neq 0$. In this lecture, we consider $\mathbb{P}_n$ as a subspace of the linear space $\mathcal{C}_{[a,b]}$ of continuous real (or complex) valued functions on the interval $[a,b]$, $a < b$. For $m \in \mathbb{N}$, we denote as $\mathcal{C}^m[a,b]$ the linear space of $m$ times continuously differentiable real (or complex) functions.

**Theorem 2.** *For $n \in \mathbb{N} \cap \{0\}$, each polynomial in $\mathbb{P}_n$ that has more than $n$ (complex) zeros , where each zero is counted repeatedly according to its multiplicity, must vanish (i.e. all of its coefficients must be identically zero)*

*Proof.* To be done □

**Theorem 3.** *The monomials $\left\{x^k\right\}_{k\geq 0}$ are linearly independent*

The linear independence of the monomials $1, x, x^2, \ldots$ implies that those monomials can be used to define a basis of $\mathbb{P}_n$ and that $\mathbb{P}_n$ has dimension $n+1$

Lagrange's solution to the polynomial interpolation problem is based on the following elementary functions. For a set of $n+1$ arbitrary support points $x_i$, $i = 0, \ldots, n$ with $x_i \neq x_k$ for $i \neq k$, there is a unique polynomial $\ell_j(x)$ (known as Lagrange interpolation polynomial) satisfying

$$\ell_j(x_k) = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{9}$$

The Lagrange polynomials are equivalently defined as

$$\ell_j(x) = \frac{(x-x_0)\dots(x-x_{j-1})(x-x_{j+1})\dots(x-x_n)}{(x_j-x_0)(x_j-x_1)\dots(x_j-x_{j-1})(x_j-x_{j+1})\dots(x_j-x_n)} \tag{10}$$

$$= \prod_{\substack{i=0\\ i\neq j}}^{n} \frac{(x-x_i)}{(x_j-x_i)}, \quad j=0,\dots,n \tag{11}$$

**Theorem 4** (Lagrange interpolation Theorem)**.** *Given $n+1$ distinct points $x_0, x_1, \dots, x_n \in [a,b]$, and $n+1$ values $f_0, f_1, \dots, f_n \in \mathbb{R}$, there exists a unique polynomial $p_n(x) \in \mathbb{P}_n$ with the property*

$$p_n(x_j) = f_j, \quad j=0,\dots n$$

*In the Lagrange representation, this interpolation polynomial is given by*

$$p_n(x) = \sum_{j=0}^{n} f_j \ell_j(x)$$

*where the Lagrange polynomials $\ell_j(x)$ are defined as in* (9)-(11).

Although the values of the function $f$ and those of its Lagrange interpolation coincide at the interpolation points $x_k$, $f(x)$ can be quite different from the interpolating polynomial for $x \neq x_k$. In particular, it is natural to wonder how large the difference $f(x)-p_n(x)$ is when $x \neq x_k$, $k=0,\dots,n$. Assuming that the function $f$ is sufficiently smooth, an estimate of the size of the interpolation error $f(x)-p_n(x)$ is given by the following theorem

**Theorem 5.** *Suppose that $n \geq 0$ and that $f$ is a real-valued function, defined and continuous on the closed real interval $[a,b]$, such that the derivative of $f$ of order $n+1$ exists and is continuous on $[a,b]$. Then, given that $x \in [a,b]$, there exists $\xi = \xi(x)$ in $(a,b)$ such that*

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x) \tag{12}$$

*where*

$$\pi_{n+1}(x) = (x-x_0)\dots(x-x_n)$$

*From this we thus have,*

$$|f(x)-p_n(x)| \leq \frac{M_{n+1}}{(n+1)!}\,|\pi_{n+1}(x)| \tag{13}$$

*where*

$$M_{n+1} = \sup_{\xi\in[a,b]} \left| f^{(n+1)}(\xi) \right|$$

*Proof of the interpolation theorem.* At the interpolation points, i.e. when $x = x_i$, for $i = 0, \dots n$, both the interpolating polynomial and the original function are zero and the equality is trivially satisfied. Let us now focus on $x \in [a, b]$ with $x \neq x_i$ and let us introduce the function $\varphi(y)$ as

$$\varphi(y) \ : \ u \mapsto \varphi(y) = f(y) - p_n(y) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(y). \tag{14}$$

The function $\varphi(y)$ vanishes at $y = x_i$, $i = 0, \dots, n$ as well as $y = x$ ($n + 2$ distinct points). Using Rolle's theorem, the derivative of $\varphi(y)$, $\varphi'(y)$ must therefore vanish at $n + 1$ points located in between the zeros of $\varphi(y)$. As $\varphi'(y)$ vanishes at $n + 1$ points, $\varphi''(y)$ vanishes at $n$ points and we therefore have

$$f''(\xi) - p_n''(\xi) = \frac{(f(x) - p_n(x))}{\pi_{n+1}(x)} \pi_{n+1}''(\xi)$$

continuing like this until $\varphi^{n+1}(\xi)$, we can find a $\xi \in (a, b)$ such that $\varphi^{(n+1)}(\xi) = 0$ and we can write

$$\varphi^{(n+1)}(\xi) = 0 = f^{(n+1)}(\xi) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} (n + 1)! \tag{15}$$

□

# Hermite interpolation

The idea of Lagrange interpolation can be generalized in various ways. A popular alternative consists in requiring the polynomial $p$ to take given function values and derivative values at the interpolation points. For a set of distinct interpolation points $x_k$, $k = 0, \dots, n$ and two sets of real numbers $y_k$, $k = 0, \dots, n$ and $z_k$, $k = 0, \dots, n$ with $n \geq 0$, we want to find a polynomial $p_{2n+1} \in \mathbb{P}_{2n+1}$ such that

$$p_{2n+1}(x_k) = y_k, \quad p'_{2n+1}(x_k) = z_k, \quad k = 0, \dots, n$$

The construction is similar to that of the Lagrange interpolation polynomials but now requires two sets of polynomials $H_\ell$ and $K_\ell$ with $\ell = 0, \dots, n$ which are defined as follows

**Theorem 6** (Hermite interpolation theorem)**.** *Let $n \geq 0$ and suppose that $x_k$ $k = 0, \dots, n$ are distinct real numbers. Then, given two sets of real numbers $y_k$, $k = 0, \dots, n$ and $z_k$, $k = 0, \dots, n$, there is a unique polynomial $p_{2n+1}$ in $\mathbb{P}_{2n+1}$ such that*

$$p_{2n+1}(x_k) = y_k, \quad p'_{2n+1}(x_k) = z_k, \quad k = 0, \dots, n$$

*Proof.* We keep the notation $\ell_j$ to denote the Lagrange polynomials and we further introduce two sets of auxiliary polynomials $H_k$ and $K_k$, $k = 0, 1, \ldots, n$

$$H_k(x) = [\ell_k(x)]^2 \left(1 - 2\ell_k'(x_k)(x - x_k)\right) \tag{16}$$

$$K_k(x) = [\ell_k(x)]^2 (x - x_k) \tag{17}$$

where $\ell_k$ is used to denote the usual Lagrange polynomial such as defined in (9) or (11).

From our definition of the $\ell_j(x)$, it is easy to see that

$$H_k(x_i) = K_k(x_i) = 0 \tag{18}$$

$$H_k'(x_i) = K_k'(x_i) = 0 \tag{19}$$

for $x_i \neq x_k$. Moreover,

$$H_k(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

I.e. (19) follows from

$$H_k'(x) = 2\ell_k\ell_k' - 4\ell_k\ell_k'\ell_k'(x_k)(x - x_k) - 2(\ell_k)^2\ell_k'(x_k)$$

Finally note that from (17), $K_k(x_k) = 0$ and $K_k'(x_k) = 2\ell_k\ell_k'(x - x_k) + \ell_k^2(x)$ which in particular gives $K_k'(x_k) = 1$. Grouping those results, we have

$$H_k(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}, \qquad H_k'(x_i) = 0$$

$$K_k(x_i) = 0, \qquad K_k'(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

Combining those ideas, a natural way to define our interpolating polynomial is as

$$p_{2n+1}(x) = \sum_{k=0}^{n} \left[H_k(x)y_k + K_k(x)z_k\right]$$

□

The result of Theorem 6 can be summarized by the following definition

**Definition 1** (Hermite interpolation polynomial)**.** *Let $n \geq 0$ and suppose that $x_i$, $i = 0, \ldots, n$ are distinct real numbers. The polynomial $p_{2n+1}$ defined as*

$$p_{2n+1}(x) = \sum_{k=0}^{n} \left[H_k(x)y_k + K_k(x)z_k\right] \tag{20}$$

*where $H_k$ and $K_k$ are defined as in (16)-(17), is called Hermite interpolation polynomial of degree $2n + 1$ associated to the set of triples $\{(x_i, y_i, z_i), \quad i = 0, \ldots, n\}$*

Just as for the Lagrange polynomial, one can derive an error bound on the Hermite interpolation

**Theorem 7** (Hermite interpolation error)**.** *Suppose that* $n \geq 0$ *and let* $f$ *be a real valued continuous* $2n+2$ *times differentiable function on the interval* $[a,b]$ *such that* $f^{(2n+2)}$ *is continuous on* $[a,b]$*. Further let* $p_{2n+1}$ *denote the Hermite interpolation polynomial of* $f$ *defined as in* (20)*. Then for each* $x \in [a,b]$*, there exists* $\xi = \xi(x)$ *in* $(a,b)$ *such that*

$$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\left[\pi_{n+1}(x)\right]^2$$

*where* $\pi_{n+1}(x)$ *is defined as*

$$\pi_{n+1}(x) = (x-x_0)(x-x_1)\ldots(x-x_n)$$

*Moreover,*

$$|f(x) - p_{2n+1}(x)| \leq \frac{M_{2n+2}}{(2n+2)!}\left[\pi_{n+1}\right]^2$$

*where*

$$M_{2n+2} = \max_{\xi\in[a,b]}\left|f^{(2n+2)}(\xi)\right|$$

## Newton representation

Lagrange interpolation is very convenient for theoretical investigations because of its simple structure. For practical computations, it is only suitable for small $n$. For large values of $n$, the Lagrange factors become large and oscillatory. A more practical representation was obtained by Newton around 1676.

**Definition 2.** *Given $(n+1)$ distinct $x_0, \ldots, x_n \in [a, b]$ and $(n+1)$ values $y_0, y_1, \ldots, y_n \in \mathbb{R}$ the divided differences $D_j^k$ of order $k$ at the point $x_j$ are recursively defined as*

$$\begin{aligned} D_j^0 &= y_j && j = 0, \ldots, n \\ D_j^k &= \frac{D_{j+1}^{(k-1)} - D_j^{(k-1)}}{x_{j+k} - x_j} && j = 0, \ldots, n-k \end{aligned}$$

*A convenient way to represent the divided differences is as the following table*

$$\begin{array}{llll} x_0, y_0 = D_0^0 & & & \\ & D_0^0 & & \\ x_1, y_1 = D_1^0 & & D_0^2 & \\ & D_1^0 & & D_0^3 \\ x_2, y_2 = D_2^0 & & D_1^2 & \\ & D_3^0 & & \\ x_3, y_3 = D_3^0 & & & \end{array}$$

**Example 1.** *For the points $x_0 = 0, x_1 = 1, x_2 = 3, x_4 = 4$, and the values $y_0 = 0, y_1 = 2, y_2 = 8, y_4 = 9$, the table of divided differences is given by*

$$\begin{array}{llll} 0, 0 & & & \\ & 2 & & \\ 1, 2 & & 1/3 & \\ & 3 & & -1/4 \\ 3, 8 & & -2/3 & \\ & 1 & & \\ 4, 9 & & & \end{array}$$

In the Newton representation, for $n \geq 1$, the uniquely determined interpolation polynomial $p_n$ is given by

$$p_n(x) = y_0 + \sum_{k=1}^{n} D_0^k \prod_{i=0}^{k-1} (x - x_i) \tag{21}$$

*Proof.* Let $\tilde{p}_n$ denote the right-hand side of (21) and let $p_n$ denote the interpolation polynomial as obtained from the Lagrange interpolating polynomials. We will establish that $p_n = \tilde{p}_n$ by induction. For $n = 1$, we have

$$p_1 = y_0 + (x - x_0)\frac{(y_1 - y_0)}{(x_1 - x_0)}$$

let us now assume that the relation holds for the degree $n-1$ with $n \geq 2$ and let us consider the difference $d_n = p_n - \tilde{p}_n$,

$$d_n(x) = p_n - \tilde{p}_n = p_n(x) - \tilde{p}_{n-1}(x) - D_0^n \prod_{i=0}^{n-1}(x - x_i) \tag{22}$$

In the Lagrange representation, the unique interpolation polynomial can read as

$$p_n = \sum_{k=0}^{n} y_k \ell_k = \sum_{k=0}^{n} y_k \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}, \quad k = 0, \ldots, n. \tag{23}$$

On the other hand, we can show (to be done) that the divided differences satisfy the relation

$$D_j^k = \sum_{m=j}^{j+k} y_m \prod_{\substack{i=j \\ i \neq m}}^{j+k} \frac{1}{x_m - x_i}, \quad j = 0, \ldots, n-k, \quad k = 1, \ldots, n \tag{24}$$

in particular, we have

$$D_0^n = \sum_{m=0}^{n} y_m \prod_{\substack{i=0 \\ i \neq m}}^{n} \frac{1}{x_m - x_i} \tag{25}$$

which is the coefficient of $x^n$ in the Lagrange decomposition of the interpolation polynomial (23). From (22), this therefore implies that the difference $d_n$ does not have any order $n$ term. In other words, $d_n \in \mathbb{P}_{n-1}$. Note that by construction we have

$$\tilde{p}_{n-1}(x_j) = y_j = p_n(x_j), \quad j = 0, \ldots, n-1 \tag{26}$$

so that $d_n(x_j) = 0$ for $j = 0, \ldots, n-1$. Since we have just showed that the polynomial $d_n$ had degree $n-1$, this last statement implies that $d_n$ must be identically zero. □

Note that the Newton representation has the (elegant) general form

$$p_n(x) = a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) + \ldots + a_1(x - x_0) + a_0 \tag{27}$$

To conclude our discussion on interpolation, and before moving to the approximation problem, we cover one last approach at deriving the interpolation polynomial known as Neville's scheme which is particularly useful when one only needs to estimate the value of an interpolation polynomial at a point $x$ (as in this case, it can help to avoid explicitely building the polynomial)

**Theorem 8.** *Given $n+1$ distinct points $x_0, \ldots, x_n \in [a,b]$ and $n+1$ values $y_0, \ldots, y_n \in \mathbb{R}^n$, the uniquely determined interpolation polynomials $p_i^k \in \mathbb{P}_k$, $i = 0, \ldots, n-k$, $k = 0, \ldots, n$ with the interpolation property*

$$p_i^k(x_j) = y_j, \quad j = i, \ldots, i+k$$

*satisfy the recurrence relation*

$$p_i^0(x) = y_i \tag{28}$$

$$p_i^k(x) = \frac{(x-x_i)p_{i+1}^{k-1}(x) - (x-x_{i+k})p_i^{k-1}(x)}{x_{i+k}-x_i}, \quad k = 1, \ldots, n \tag{29}$$

# Best approximation

Frequently, we measure the closeness of an approximation over the interval by taking the largest deviation between the function and its approximation over the interval. Alternative definitions are also possible, such as the integral of the squared deviation.

The major results in the theory of best approximation arise from the following questions:

**a)** Under what circumstances is there a best approximation?

**b)** How can the best approximants be characterized analytically or geometrically?

**c)** How can the best approximants be computed numerically?

**d)** What are the asymptotic properties of best approximations?

## The linear approximation problem

Let $\mathcal{X}$ be a normed linear space. Select $n$ linearly independent elements $x_1, x_2, \ldots, x_n$. let $y$ be an additional element. We wish to approximate $y$ by an appropriate linear combination of the $x_1, \ldots, x_n$. The closeness of two elements will be defined as the norm of their difference. I.e. we want to make

$$\|y - (a_1x_1 + a_2x_2 + \ldots + a_nx_n)\|$$

as small as possible.

The element $y - (a_1x_1 + \ldots + a_nx_n)$ is called error or discrepancy.

**Definition 3.** *A best approximation to $y$ by a linear combination of $x_1, \dots x_n$ is an element $a_1x_1 + \dots + a_nx_n$ for which*

$$\|y - (a_1x_1 + \dots + a_nx_n)\| \le \|y - (b_1x_1 + \dots + b_nx_n)\|$$

*for every $b_1, \dots, b_n$*

The problem of finding best approximations can be pictured geometrically. The set of all linear combinations $a_1x_1 + \dots + a_nx_n$ forms a linear subspace of dimension $n$. We can picture this as a plane. The element $y$ will not, in general, lie on this plane and we would like to locate the point of the plane closest to $y$.

**Theorem 9.** *Given $y$ and $n$ linearly independent elements $x_1, \dots, x_n$, the problem of finding*

$$\min_{a_i} \|y - (a_1x_1 + a_2x_2 + \dots + a_nx_n)\| \tag{30}$$

*has a solution.*

*Proof.* Let $g(a_1, \dots, a_n)$ denote the function

$$g(a_1, \dots, a_n) \equiv \|y - (a_1x_1 + \dots + a_nx_n)\|$$

The function $g$ can be seen as a continuous function in the complex variables $a_1, a_2, \dots, a_n$. I.e we have

$$|g(a_1', a_2', \dots, a_n') - g(a_1, a_2, \dots, a_n)| \tag{31}$$
$$= |\|y - (a_1'x_1 + a_2'x_2 + \dots + a_n'x_n)\| - \|y - (a_1x_1 + a_2x_2 + \dots + a_nx_n)\|| \tag{32}$$
$$\le \|(a_1' - a_1)x_1 + (a_2' - a_2)x_2 + \dots + (a_n' - a_n)x_n\| \tag{33}$$
$$\le |a_1' - a_1| \, \|x_1\| + |a_2' - a_2| \, \|x_2\| + \dots + |a_n' - a_n| \, \|x_n\| \tag{34}$$

From this, we see that the difference (31) must be small when the difference of the $a_i's$ is small.

We can get a similar result for the function $h$ defined as

$$h(a_1, \dots, a_n) = \|a_1x_1 + \dots + a_nx_n\| \tag{35}$$

As a result both functions achieve their infimum on closed sets. We let

$$\mathcal{S} = \left\{a \in \mathbb{R}^n \mid |a_1|^2 + |a_2|^2 + \dots + |a_n|^2 = 1\right\} \tag{36}$$

$\mathcal{S}$ is a closed and bounded set and $h$ must therefore achieve a minimum on $\mathcal{S}$. The value 0 is ruled out as

$$\|a_1x_1 + a_2x_2 + \dots + a_nx_n\| = 0$$

necessarily implies $a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$ which is not possible as the $x_i$'s are assumed to be linearly independent.

Let $r = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$. We have

$$h(a_1, \ldots, a_n) = r \left\| \frac{a_1}{r} x_1 + \ldots + \frac{a_n}{r} x_n \right\| \geq \varepsilon r, \quad \varepsilon > 0$$

Similarly, we can write

$$\begin{aligned} d = \|y - (a_1 x_1 + \ldots + a_n x_n)\| &\geq \|a_1 x_1 + a_2 x_2 + \ldots + a_n x_n\| - \|y\| \\ &\geq \varepsilon r - \|y\| \end{aligned}$$

Note that $\varepsilon$ is used to denote the minimum of $h$ over the unit sphere $\mathcal{S}$ and as such does not depend on $r$. As a result, we see that, as $r \to \infty$, so does the value of $d$. Restricting ourselves to a small sphere around 0, let $\rho = \inf_{\mathbb{C}^n} d(a_1, a_2, \ldots, a_n)$ and let $R = \frac{1+\rho+\|y\|}{\varepsilon}$. If $|a_1|^2 + |a_2|^2 + \ldots + |a_n|^2 > R^2$, we have $d > \varepsilon R - \|y\| > 1 + \rho$. We can thus write

$$\inf_{\mathbb{C}^n} d(a_1, a_2, \ldots, a_n) = \inf_{B_R} d(a_1, a_2, \ldots, a_n) \tag{37}$$

where $B_R$ denotes the closed radius $R$ ball. Since $B_R$ is closed and $d$ is continuous, the infimum on the RHS of (37) is a minimum which concludes the proof. □

The following corollary focuses on the setting of univariate polynomial approximation.

**Corollary 9.1.** *Let $f \in C[a, b]$ and $n$ be a fixed integer. The problem of finding*

$$\min_{a_0, a_1, \ldots, a_n} \max_{a \leq x \leq b} |f(x) - (a_0 + a_1 x + \ldots + a_n x^n)| \tag{38}$$

*has a solution.*

Since the polynomial of best approximation corresponds to minimizing the maximum value of the error $f(x) - p_n(x)$, it is often referred to as the minimax polynomial.

As we will see, although finding minimax polynomial is in general hard, in some particular settings, this solution can be shown to be unique and built explicitly by means of a family of polynomials known as the Chebyshev approximation polynomials. Moreover, we will see that the zeros of the Chebyshev polynomials, when used as interpolation points achieve the minimum of the interpolation error (13).

## Chebyshev polynomials

When discussing the general setting of polynomial interpolation, we derived an upper bound on the deviation $|f(x) - p_n(x)|$ of the form

$$R_n \equiv |f(x) - p_n(x)| \leq \max_{\xi \in [a,b]} \left| f^{(n+1)}(\xi) \right| \frac{\pi_{n+1}(x)}{(n+1)!} \tag{39}$$

where $\pi_{n+1}(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$. It is important to keep in mind that the RHS in (39) is ony an upper bound and this bound is by no means guaranteed to be tight. I.e. in many cases, the predicted error will be far greater than the actual error.

Moreover, nothing guarantees that the maximum value of $|f(x)-p_n(x)|$ will match the actual minimax bound (39) (I.e. the interpolation polynomial $p_n$ does not necessarily match the minimax polynomial achieving the minimum in (38)). But that is another story.

The error estimate for polynomial interpolation (39) splits into two main contributions. The first factor $\max_{\xi\in[a,b]} f^{(n+1)}(\xi)$ depends only on the function being interpolated but is independent on the manner in which the interpolation is carried out. The second part,

$$\frac{1}{(n+1)}|x-x_0||x-x_1|\ldots|x-x_n|$$

is independent of the function.

Although we cannot optimize the first factor as this factor will differ for every new function, we can however try to optimize the second one. Controlling that second factor will imply a small (i.e. controlled) interpolation error.

In order to control this second factor, the only freedom we have is in the selection of the interpolation points $x_0, x_1, \ldots, x_n$. It turns out that the set of interpolation points that achieve the minimum of $(x-x_0)(x-x_1)\ldots(x-x_n)$ reveals particularly interesting for a number of questions in the theory of interpolation.

It also turns out that those interpolation points can be defined as the zeros of particular polynomials known as Chebyshev polynomials which we now introduce.

Recall that from de Moivre's formula $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. Assuming $\theta\in[0,\pi]$ and letting $x=\cos\theta$, $\sin\theta=\sqrt{1-x^2}$, we can write

$$\cos n\theta + i\sin n\theta = (x+i\sqrt{1-x^2})^n$$

Expanding this expression using the binomial theorem, and taking the real part, we get

$$\cos n\theta = \sum_{k=0}^{\lfloor n/2\rfloor} x^{n-2k}(1-x^2)^k\binom{n}{2k}$$

which is a degree $n$ polynomial in $x=\cos\theta$.

**Definition 4.** *The Chebyshev polynomial of degree n is defined as*

$$T_n(x) = \cos n\theta = \cos(n\arccos x) = x^n + \binom{n}{2}x^{n-2} + \ldots, \quad n=0,1,\ldots \tag{40}$$

It is relatively easy to compute closed form expressions for the first few Chebyshev polynomials

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x
\end{aligned}$$

The Chebyshev polynomials happen to satisfy a recurrence relation

**Theorem 10.** *Let $T_n(x)$ be defined as in Definition 4, we have*

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, \ldots \tag{41}$$

*Proof.* Recall that

$$\begin{aligned}
\cos(n+1)\theta &= \cos n\theta \cos\theta - \sin n\theta \sin\theta \\
\cos(n-1)\theta &= \cos n\theta \cos\theta + \sin n\theta \sin\theta
\end{aligned}$$

Combining, we get

$$\cos(n+1)\theta = 2\cos n\theta \cos\theta - \cos(n-1)\theta$$

To get the conclusion, simply substitute $\theta = \arccos x$ and use $\cos n \arccos x = T_n(x)$. □

We can now characterize the zeros of $T_n(x)$.

**Theorem 11.** *Let $T_n(x)$ be defined as in Definition 4. The polynomial $T_n(x)$ has simple zeros at the $n$ points*

$$x_k = \cos\frac{2k-1}{2n}\pi, \quad k = 1, \ldots, n \tag{42}$$

*On the closed interval $[-1, 1]$, $T_n(x)$ has extreme values at the $n+1$ points*

$$x'_k = \cos\frac{2k}{2n}\pi, \quad k = 0, 1, \ldots, n \tag{43}$$

*where it assumes the alternating values $(-1)^k$.*

*Proof.* Note that we have

$$T_n(x_k) = \cos\left(n \arccos\left(\cos\frac{2k-1}{2n}\pi\right)\right) = \cos\left(\frac{(2k-1)\pi}{2}\right) = 0$$

Similarly,

$$\begin{aligned} T_n'(x) &= \frac{d}{dx}\cos(n\arccos x) \\ &= -n\sin(n\arccos x)\frac{d}{dx}\arccos x \\ &= n\sin(n\arccos x)\frac{1}{\sqrt{1-x^2}} \end{aligned}$$

From this, one can check that

$$\begin{aligned} T_n'(x_k) &= \frac{n}{\sqrt{1-x_k^2}}\sin(n\arccos x_k) = \frac{n}{\sqrt{1-x_k^2}}\sin\left(\frac{2k-1}{2}\pi\right) \\ &= \pm\frac{n}{\sqrt{1-x_k^2}} \neq 0 \end{aligned}$$

This last relation shows that the zeros $x_k$ of $T_n(x)$ are simple. We can also check that

$$\begin{aligned} T_n'(x_k') &= \frac{n}{\sqrt{1-x_k^2}}\sin\left(n\arccos x_k'\right) \\ &= \frac{n}{\sqrt{1-x_k^2}}\sin\left(\frac{2k\pi}{2}\right) = 0 \end{aligned}$$

Finally,

$$T_n(x_k') = \cos\left(\pi\right) = (-1)^k$$

since $|T_n(x)| = |\cos(n\arccos x)| \leq 1$, this confirms that the points $x_k'$ are extreme points of $T_n(x)$. □

We are now ready to tackle our original objective corresponding to finding the interpolation points that minimize the polynomial $\pi_{n+1}(x)$ on the RHS of (39).

**Theorem 12.** *Let $\tilde{\mathbb{P}}_n$ denote the class of all polynomials of degree $n$ with leading coefficient 1. Let $\tilde{T}_n = T_n/2^{n-1}$ (The Chebyshev polynomial normalized so that its leading coefficient equals 1) Then for any $\tilde{p} \in \tilde{\mathbb{P}}_n$, we have*

$$\max_{-1\leq x\leq 1}\left|\tilde{T}_n(x)\right| \leq \max_{-1\leq x\leq 1}|\tilde{p}(x)| \tag{44}$$

*Proof.* Clearly, from the definition of $T_n(x)$, the polynomial $|\tilde{T}_n(x)|$ achieves its maximum $1/2^{n-1}$ at the $n+1$ points $x_k' = \cos\frac{k\pi}{n}$, $k = 0, 1, \ldots, n$. Suppose there exists $\tilde{p} \in \tilde{\mathbb{P}}_n$ s.t.

$$\max_{x\in[-1,1]}|\tilde{p}(x)| < \frac{1}{2^{n-1}} \tag{45}$$

Let $q(x) \in \mathbb{P}_{n-1}$ be defined as $q(x) = \tilde{T}_n(x) - \tilde{p}(x)$. Note that we have

$$q(x'_k) = \tilde{T}_n(x'_k) - \tilde{p}(x'_k) \tag{46}$$
$$= \frac{1}{2^{n-1}}(-1)^k - \tilde{p}(x'_k) \tag{47}$$

From this in particular, we see that the polynomial $q$ takes alternate values at the $n+1$ points $x'_k$. $q$ must therefore have at least $n$ distinct zeros which is contraction as $q$ is in $\tilde{P}_{n-1}$. □

From the result of Theorem 12, we see that

$$|(x-x_0)(x-x_1)\dots(x-x_n)| \geq \max_{x\in[-1,1]} \left|\tilde{T}_n(x)\right| = \frac{1}{2^{n-1}}$$

As a result, if we choose our interpolation points to be the roots of $T_{n+1}(x)$, our upper bound on the interpolation error (39) satisfies

$$R_n(f) \equiv |f(x) - p_n(x)| \leq \max_{\xi\in[a,b]} \left|f^{(n+1)}(\xi)\right| \frac{\left|\tilde{T}_n(x)\right|}{(n+1)!} \tag{48}$$
$$\leq \frac{1}{2^{n-1}(n+1)!} \max_{\xi\in[a,b]} \left|f^{(n+1)}(\xi)\right| \tag{49}$$

In particular, provided that $\max_{\xi\in[a,b]} |f^{(k)}(\xi)|$ is bounded by a constant $C$ for all $k$, the upper bound (49) will vanish as $n \to \infty$.

## Minimax Approximation

Now that we have introduced the Chebyshev polynomials, we can go back to our original problem of finding the minimax polynomial $p_n \in \mathbb{P}_n$, that is to say the polynomial $p_n$ (not necessarily interpolating $f$) that achieves the smallest error in the $\ell_\infty$ norm,

$$\|f(x) - p_n^*(x)\|_\infty = \min_{p\in\mathbb{P}_n} \|f-p\|_\infty \tag{50}$$

There are very few functions for which it is possible to write down a simple closed form for the minimax polynomial. One setting in which such a polynomial admits a simple expression is the approximation of a power of $x$ by a polynomial of lower degree. The minimax approximation in this case, can be obtained from the Chebyshev polynomials. This idea is summarized by the following theorem

**Theorem 13.** *Let $n \geq 0$. The polynomial $p_n \in \mathbb{P}_n$ defined by*

$$p_n(x) = x^{n+1} - \frac{T_{n+1}(x)}{2^n} \tag{51}$$

*is the minimax approximation of degree $n$ to the function $x \mapsto x^{n+1}$ on the interval $[-1,1]$.*

*Proof.* To be done. □

## Trigonometric interpolation

In engineering and the natural sciences, it is also common to encounter periodic functions, i.e. functions with the property $f(t+\tau) = f(t)$, $t \in \mathbb{R}$ for some $\tau \in \mathbb{R}$. Aside from the more traditional setting, the family of periodic functions also includes functions defined on closed planes and space curves. Polynomial interpolation as introduced above is not appropriate for periodic functions as polynomials are not periodic. An alternative is to turn to trigonometric polynomials which were first used independently by Clairaut (1759) and Lagrange (1762). We will get back to those notions when discussing the Fourier series and Fourier transform.

**Definition 5** (General trigonometric polynomial)**.** *For $n \in \mathbb{N}$, we denote by $\mathbb{T}_n$ the linear space of trigonometric polynomials*

$$q(t) = \sum_{k=0}^{n} a_k \cos kt + \sum_{k=1}^{n} b_k \sin kt$$

*with real (or complex) coefficients $a_0, \dots, a_n$ and $b_1, \dots, b_n$. A trigonometric polynomial $q \in \mathbb{T}_n$ is said to be of degree $n$ if $|a_n| + |b_n| > 0$*

Note that in the definition above, we restrict our attention to polynomials of period $T = 2\pi$. We can of course similarly define polynomials of arbitrary period $T$ as

$$q_T(t) = \sum_{k=0}^{n} a_k \cos \frac{2\pi kt}{T} + \sum_{k=1}^{n} b_k \sin \frac{2\pi kt}{T}$$

Just as for regular algebraic polynomials, one can derive a relation between the degree of a trigonometric polynomial and the maximum number of distinct zeros of this polynomial.

**Theorem 14.** *A trigonometric polynomial in $\mathbb{T}_n$ that has more than $2n$ distinct zeros in the interval $[0, 2\pi)$ must vanish identically (i.e. all its coefficients must be equal to zero)*

*Proof.* Let

$$q(t) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos kt + b_k \sin kt]$$

setting $b_0 = 0$ and $\gamma_k = \frac{1}{2}(a_k - ib_k)$, $\gamma_{-k} = \frac{1}{2}(a_k + ib_k)$ and using Euler's formula, $e^{it} = \cos t + i \sin t$, we can rewrite our polynomial $q(t)$ in complex form as

$$q(t) = \sum_{k=-n}^{n} \gamma_k e^{ikt}$$

letting $z = e^{it}$ and using $p(z)$ to denote the polynomial (in $z$)

$$p(z) = \sum_{k=-n}^{n} \gamma_k z^{n+k}$$

we get $q(t) = z^{-n} p(z)$. If the trigonometric polynomial $q(t)$ has more than $2n$ distinct zeros $t_\ell$, $\ell = 1, \ldots, L > 2n$ on the $[0, 2\pi)$ interval, then to each zero of $q(t)$, one can associate a distinct zero $z_\ell = e^{it_\ell}$ of $p(z)$ on the unit circle. □

A natural consequence of the connection between the maximum number of distinct zeros and the vanishing of trigonometric polynomials is the linear independence of the trigonometric functions in $\mathbb{T}_n$ and even in $\mathcal{C}[0, 2\pi]$. This idea can be summarized by the following theorem,

**Theorem 15.** *The cosine functions* $\cos kt$, $k = 0, 1, \ldots, n$ *and the sine functions* $\sin kt$, $k = 1, \ldots, n$ *are linearly independent in the function space* $\mathcal{C}[0, 2\pi]$

*Proof.* Assume that one of the $\{\cos kt\}_{k \geq 0} \cup \{\sin kt\}_{k \geq 0}$ can be expressed as a linear combination of the other trigonometric functions. In this case, there exists coefficients $a_k \cup b_k$ such that

$$\sum_{k=0}^{n} a_k \cos kt + \sum_{k=0}^{n} b_k \sin kt = 0, \quad \forall t \in [0, 2\pi)$$

This relation in particular implies that the corresponding polynomial has more than $2n$ zeros. From the relation between the degree of the polynomial and the maximum number of distinct zeros, we deduce that all the coefficients must vanish. □

From the proof of Theorem (15), we can derive a result similar to the interpolating properties of algebraic polynomials

**Theorem 16.** *Given $2n+1$ distinct points $t_0, \ldots, t_{2n} \in [0, 2\pi]$ and $2n+1$ distinct values $y_0, \ldots, y_{2n} \in \mathbb{R}$, there exists a uniquely determined trigonometric polynomial $q_n \in \mathbb{T}_n$ with the property*

$$q_n(t_j) = y_j, \quad j = 0, \ldots 2n$$

*In the Lagrange representation, this trigonometric interpolation polynomial is given by*

$$q_n = \sum_{k=0}^{2n} y_k \ell_k$$

*with Lagrange factors*

$$\ell_k(t) = \prod_{\substack{i=0 \\ i \neq k}}^{2n} \frac{\sin \frac{t - t_i}{2}}{\sin \frac{t_k - t_i}{2}}, \quad k = 0, \ldots, 2n$$

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