Numerical Analysis
Lecture 4

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January 2022

This note was written as part of the series of lectures on Numerical Analysis delivered at ULCO in 2022-2023. The version is temporary. Please direct any comments or questions to augustin.cosse@univ-littoral.fr.

The Fourier series

Introducing the notion of Fourier series and Fourier transform first requires to discuss the notion of periodic and orthogonal functions.

Definition 1. *A function $f(x)$ of one variable x is said to be periodic with period $T > 0$ if the domain $\mathcal{D}(f)$ of f contains $x + T$ whenever it contains x and if for every $x \in \mathcal{D}(f)$, one has $f(x + T) = f(x)$.*

The Fourier series and its extension to non periodic functions, the Fourier transform follow from the orthogonality of trigonometric functions

Definition 2. *Two functions u and v are said to be orthogonal on $[a, b]$ if the product uv is integrable and*

$$\int_a^b u(x)\overline{v}(x)\;dx = 0$$

where $\overline{v}$ indicates the complex conjugate. A set of functions is said to be mutually orthogonal if each distinct pair in the set is orthogonal on $[a, b]$

Lemma 1. *The functions*

$$1, \sin\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}, \quad m = 1, 2, \dots$$

form a mutually orthogonal set on the interval $[-L, L]$ *as well as on every interval* $[a, a + 2L]$. *In fact*

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases} \tag{1}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = 0 \tag{2}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases} \tag{3}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} = \int_{-L}^{L} \cos\frac{m\pi x}{L}\, dx = 0 \tag{4}$$

Proof. We do the proof for (3), the other three relations follow the exact same idea. First recall that we have

$$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$

for $m \neq n$, we can thus write

$$\begin{aligned}
\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx &= \frac{1}{2}\int_{-L}^{L} \left(\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right)\, dx \\
&= \frac{1}{2}\left|\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{\frac{(m-n)\pi}{L}}\right|_{-L}^{L} - \frac{1}{2}\left|\frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{\frac{(m+n)\pi}{L}}\right|_{-L}^{L} \\
&= 0
\end{aligned}$$

When $m = n$, a similar reasoning gives

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L} \left(1 - \cos\left(\frac{2m\pi x}{L}\right)\right)\, dx = L$$

□

The above lemma also holds for the functions $e^{\frac{in\pi x}{L}}$, $n = 0, \pm 1, \pm 2, \dots$

$$\int_{-L}^{L} e^{\frac{in\pi x}{L}} e^{-\frac{-im\pi x}{L}}\, dx = \begin{cases} 0 & n \neq m \\ 2L & n = m \end{cases}$$

To introduce the notion of Fourier series, we start by considering a general series of the form

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \Big(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \Big) \tag{5}$$

If such a series converges, the corresponding limit must be $2L$ periodic.

Let us denote this limit as $f(x)$. I.e. let us assume that there exists $f(x)$ with

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \Big(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \Big) \tag{6}$$

To determine the coefficients a_m, b_m, we proceed as follows. Assuming that the integration can be carried out term by term (which holds for example if $\sum_{m=1}^{\infty}(|a_m| + |b_m|) < \infty$), we multiply the series (6) by any of the trigonometric functions, e.g. $\cos \frac{n\pi x}{L}$ and integrate to get

$$\begin{aligned} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = \frac{a_0}{2} \int_{-L}^{L} \cos \frac{n\pi x}{L}\, dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx \\ + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx \end{aligned}$$

It follows from the orthogonality of the trigonometric functions that

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \tag{7}$$

A similar argument holds for b_n. Using the orthogonality of the $\sin \frac{n\pi x}{L}$, we get

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \tag{8}$$

Finally, for the constant coefficient a_0, using the relations

$$\int_{-L}^{L} \sin \frac{m\pi x}{L}\, dx = \int_{-L}^{L} \cos \frac{m\pi x}{L}\, dx = 0$$

we get

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx \tag{9}$$

Note that so far, we haven't discussed convergence of the series. We have only shown that if the series admits a limit $f(x)$, such that

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L}$$

necessarily, the coefficients a_m, b_m must obey the above relations. This connection leads to the notion of Fourier series.

Definition 3. *let f be integrable (not necessarily periodic), on the interval $[-L, L]$. The Fourier series of f is the trigonometric series (41) where the coefficients a_0, a_m, b_m are given by (7), (8) and (9) respectively. In that case, we write*

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right), \quad \text{on } [-L, L] \tag{10}$$

Note that the above definition does not imply that the series converges to f or that f is periodic (it is the limit of the series that has to be periodic and this limit matches f only on $[-L, L]$).

If the function f is even on $[-L, L]$, the coefficients b_m must vanish and the series reduces to

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi x}{L}$$

Similarly, if the function f is odd, the same holds for the coefficients a_m so that the series in this case reduces to

$$f(x) \sim \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{L}$$

The interesting aspect of the Fourier series is that even if the function is not periodic but only defined on a finite interval, it remains possible to find an approximation of this function by (i) extending it into a periodic function and (ii) computing the series of the extension. Note that there are multiple options when considering extensions:

- Even Extensions. In this case, for a function f defined on an asymmetric interval $[0, L]$ with arbitrary $L > 0$, we can define a function $g(x)$ on the interval $[-L, L]$ as

  $$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L < x < 0 \end{cases}$$

  In this case, $g(x)$ can be encoded by a cosine series which also represents f on $[0, L]$

- Odd extension. In this case, we can define a function $h(x)$ on the interval $[-L, L]$ as

  $$h(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

  Then $h(x)$ can be encoded by a sine series whose restriction on $[0, L]$ can then be used as an approximation for f

Finally note that the series can always be written in complex form

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

for which, since the function $f(x)$ is real, the imaginary part should vanish. By comparison with the decomposition (10), we get

$$\begin{aligned} a_n &= c_n + c_{-n}, \quad n = 1, 2, \dots \\ a_0 &= 2c_0 \\ b_n &= i(c_n + c_{-n}) \quad n = 1, 2, \dots \end{aligned}$$

Inversely, we can express the coefficients of the complex series from the coefficients of the sine/cosine series as

$$c_n = \begin{cases} \frac{a_n}{2} + \frac{b_n}{2i} & n = 1, 2 \dots \\ \frac{a_0}{2} & n = 0 \\ \frac{a_{-n}}{2} - \frac{b_{-n}}{2i} & n = -1, -2, \dots \end{cases}$$

Finally, the orthogonality of the trigonometric functions implies (just as for the a_n, b_n and a_0),

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i\frac{n\pi x}{L}}\, dx$$

for $n = 0, \pm 1, \pm 2, \dots$. We call c_n, the n^{th} Fourier coefficient of f. It can be checked that $c_n = \bar{c}_{-n}$

The Fourier transform

There are usually two ways of approaching the Fourier transform. One is to define Fourier transforms by increasing the period of the function f. This leads to the Fourier series becoming, in the limit of an infinite period, an integral. The other approach is to view Fourier transforms as a particular instance of an integral transform whose properties can be compared with the properties of the Laplace transform. In these notes, we follow the first approach.

Definition 4. *Let f be a function defined for all $x \in \mathbb{R}$ with values in $\mathbb{C}$. The Fourier transform is defined as the mapping $\mathcal{F} : \mathbb{R} \mapsto \mathcal{F}\{f\}(\omega)$ defined as*

$$F(\omega) = \mathcal{F}\{f\}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \qquad (11)$$

Of course, there are some functions for which the integral does not exist. Now provided that $F(\omega)$ is well defined, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} \, d\omega \tag{12}$$

which is known as the inverse Fourier Transform.

From the Fourier series to the Fourier transform

To derive the Fourier transform from the Fourier series formally, we need the notion of Schwartz class $\mathcal{S}$.

Definition 5. *A function f belongs to the Schwartz class (i.e. $f \in \mathcal{S}(\mathbb{R}^n)$) if for all $m, N > 0$ there exists a constant $C(N, m)$ such that*

$$\left| \frac{d^m g}{dx^m} \right| \leq C(m, N) \left(\frac{1}{(1 + |x|^2)^{1/2}} \right)^N$$

Note that if $f \in \mathcal{S}(\mathbb{R}^n)$, we necessarily have

$$|f(x)| \leq c_m \, (1 + |x|)^{-m}$$

for every $m \in \mathbb{N}$.

For a function in the Schwartz class, all the derivatives are garanteed to vanish at infinity.

Given a general function f, since the Fourier series is defined for periodic functions only, we first need to make f periodic. For this we introduce the function $\chi(t)$ defined as

$$\chi(t) = \begin{cases} 1 & -1 \leq t \leq 1 \\ 0 & |t| > 1 \end{cases}$$

For any $T \in \mathbb{R}$, we can then introduce the truncated function $\chi\left(\frac{2t}{T}\right) f(t)$ which is equal to $f(t)$ on $[-T/2, T/2]$ and zero otherwise and we consider the T-periodic function g defined as

$$g(t) = \begin{cases} f(t) & \text{on } [-T/2, T/2] \\ \\ f(t - nT) & \text{on } (2n-1)\frac{T}{2} < |t| < (2n+1)\frac{T}{2}, \quad n \in \mathbb{Z}, n \neq 0 \end{cases}$$

(I.e. g is just the repetition of the truncation $\chi\left(\frac{2t}{T}\right) f(t)$ on the real line). Since the function g is periodic, we can compute its Fourier series (note that here we take T to be the period),

$$g(t) = \sum_{n=-\infty}^{\infty} g_n e^{\frac{2\pi i n t}{T}}$$

where the coefficients g_n are defined as

$$g_n = \int_{-T/2}^{T/2} g(t') e^{-\frac{2\pi i n t'}{T}} \, dt'$$

We will see that for T large enough, the Fourier coefficients of g are close to those of f, i.e.

$$\begin{aligned} g_n &= \frac{1}{T} \int_{-T/2}^{T/2} g(t') e^{-\frac{2\pi i n t'}{T}} \, dt' \\ &= \frac{1}{T} \int_{-T/2}^{T/2} f(t') \, \chi\left(\frac{2t'}{T}\right) e^{-\frac{2\pi i n t'}{T}} \, dt' \\ &\approx \frac{1}{T} \int_{-\infty}^{\infty} f(t') e^{-\frac{2\pi i n t'}{T}} \, dt' \end{aligned}$$

which we denote as $\hat{f}\left(\frac{2\pi n}{T}\right)$. From this, we then have

$$g(t) \approx \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{2\pi n}{T}\right) e^{\frac{2\pi i n}{T} t} \tag{13}$$

In particular, for $t \in [-T/2, T/2]$, if (13) holds, for T large enough, we can thus hope to have

$$f(t) \approx \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{2\pi n}{T}\right) e^{\frac{2\pi i n}{T} t} \tag{14}$$

The expression (14) is a Riemann sum, whose particularity lies in the fact that the limit for $T \to \infty$ corresponds to the definition of the integral. Applying this idea, we should then be able to write

$$f(t) = \lim_{T\to\infty} g(t) \approx \lim_{T\to\infty} \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{2\pi n}{T}\right) e^{\frac{2\pi i n}{T} t} \tag{15}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega \tag{16}$$

We now provide the formal derivation (see [5] for more details). We first note that for a function in the Schwartz class, we have

$$\left| \int_{-\infty}^{\infty} f(t) e^{-\frac{2\pi i n t}{T}} \, dt - \int_{-T/2}^{T/2} g(t) e^{-\frac{2\pi i n t}{T}} \, dt \right| \tag{17}$$

$$= \left| \int_{-\infty}^{\infty} f(t) \left(1 - \chi\left(\frac{2t}{T}\right)\right) e^{-\frac{2\pi i n t}{T}} \, dt \right| \tag{18}$$

$$\leq \left(\frac{T}{2\pi i n}\right)^r \left| \int_{-\infty}^{\infty} \frac{d}{dr} \left\{ f(t) \left(1 - \chi\left(\frac{2t}{T}\right)\right) \right\} e^{-\frac{2\pi i n t}{T}} \, dt \right| \tag{19}$$

where the last line follows from an integration by parts, setting $f' = e^{-\frac{2\pi i n t}{T}}$, $g = f(t)\left(1 - \chi\left(\frac{2t}{T}\right)\right)$ and noting that since f is in the Schwartz space, the first $|fg|$ primitive vanishes when taken in the limit $t \to \pm\infty$, i.e.

$$|f(t)| \leq \frac{C}{(1+|t|)^\alpha}, \quad \text{for every } \alpha > 0$$

Substituting this in our sequence of approximations, we get

$$
\begin{aligned}
g(t) &= \sum_{n=-\infty}^{\infty} g_n e^{\frac{2\pi i n t}{T}} && (20)\\
&= \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t')\,\chi\left(\frac{2t'}{T}\right) e^{-\frac{2\pi i n t'}{T}}\;dt' && (21)\\
&= \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-\infty}^{\infty} f(t') e^{-\frac{2\pi i n t'}{T}}\;dt' && (22)\\
&+ \sum_{n=-\infty}^{\infty}\left(\frac{1}{T}\int_{-T/2}^{T/2} f(t')\,\chi\left(\frac{2t'}{T}\right) e^{-\frac{2\pi i n t'}{T}} - \frac{1}{T}\int_{-\infty}^{\infty} f(t') e^{-\frac{2\pi i n t'}{T}}\;dt'\right) && (23)\\
&= \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-\infty}^{\infty} f(t')\, e^{-\frac{2\pi i n t'}{T}}\;dt' + \sum_{n=-\infty}^{\infty}\frac{e_n}{T} && (24)
\end{aligned}
$$

where e_n can be bounded as above

$$
\begin{aligned}
|e_n| &\leq \left(\frac{T}{2\pi i n}\right)^r \left|\int_{-\infty}^{\infty} \frac{d^r}{d(t')^r}\left(f(t')\left(1-\chi\left(\frac{2t'}{T}\right)\right)\right) e^{-\frac{2\pi i t'}{T}}\;dt'\right| && (25)\\
&\leq \left(\frac{T}{2\pi i n}\right)^r \left|\int_{-\infty}^{\infty} \frac{d^r}{d(t')^r}\left(f(t)\left(1-\chi\left(\frac{2t'}{T}\right)\right)\right)\;dt\right| && (26)
\end{aligned}
$$

Again using the fact that f is in the Schwartz class, and since $\left(\chi\left(\frac{2t}{T}\right)-1\right)$ vanishes over $[-T/2, T/2]$, we get

$$\left|\frac{d^r f}{dt^r}\right| \leq C\left(\frac{1}{1+|t'|^2}\right)^{N/2}$$

Substituting this in (26), and noting that

$$\int_{T/2}^{\infty} C\left(\frac{1}{1+|t|^2}\right)^{N/2}\;dt \leq \int_{T/2}^{\infty} C\left(\frac{1}{|t|}\right)^{N}\;dt \leq C'\left(\frac{1}{|T/2|}\right)^{N+1}$$

we get

$$e_n \leq \left(\frac{T}{2\pi i n}\right)^r C'\left(\frac{2}{T}\right)^{N+1}.$$

In particular, taking $r = 2$ we can write

$$e_n \leq \frac{1}{n^2}\frac{C}{(T/2)^{N-1}}$$

which goes to zero as $T \to \infty$. To handle the infinite sum in (24), we simply use doinated convergence theorem.

Going back to our decomposition for $g(t)$ in (24), we now want to make the final integral of the Fourier transform appear. Although we can already see that the limit in (15) will give an integral, let us derive this formally. To do this, we consider the decomposition

$$\sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-\infty}^{\infty} f(t')\, e^{-\frac{2\pi i n t'}{T}}\, dt'\, e^{\frac{2\pi i n t}{T}} \tag{27}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-\infty}^{\infty} f(t') e^{-\frac{2\pi i n t'}{T}}\, dt'\, e^{\frac{2\pi i n t}{T}} - \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega \tag{28}$$

$$+ \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega \tag{29}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{2\pi}{T} \hat{f}\left(\frac{2\pi n}{T}\right)\, e^{\frac{2\pi i n t}{T}} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega \tag{30}$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega \tag{31}$$

where we defined $\hat{f}(\omega)$ as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t') e^{-i\omega t'}\, dt'$$

and ω is defined as the continuous limit of $\omega_n = \frac{2\pi n}{T}$

To show that (15) can indeed be written as (16), we are left with showing that the error

$$\lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{2\pi}{T} \int_{-\infty}^{\infty} f(t') e^{-\frac{2\pi i n t'}{T}}\, dt'\, e^{\frac{2\pi i n t}{T}} - \left(\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega\right)$$
$$= \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{2\pi}{T} \hat{f}\left(\frac{2\pi n}{T}\right) e^{\frac{2\pi i n t}{T}} - \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega$$

vanishes. To control this error, we write the integral as

$$\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi n/T}^{2\pi(n+1)/T} \hat{f}(\omega) e^{i\omega t}\, d\omega$$

and we control each error as

$$\left| \frac{1}{2\pi} \int_{2\pi n/T}^{2\pi(n+1)/T} \hat{f}(\omega) e^{i\omega t} d\omega - \frac{2\pi}{T} \hat{f}\left(\frac{2\pi n}{T}\right) e^{i\frac{2\pi i n}{T}} \right|$$
$$= \left| \frac{1}{2\pi} \int_{2\pi n/T}^{2\pi(n+1)/T} \frac{d}{d\omega}\left(\hat{f}(\omega) e^{i\omega t}\right) \left(2\pi \frac{n+1}{T} - \omega\right)\, d\omega \right|$$
$$\le \sup_{\omega \in \left[\frac{n}{T}, \frac{n+1}{T}\right]} \frac{1}{T^2} \frac{C}{(1+(\omega)^2)^{r/2}}$$

In the last line we again use the fact that $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$[1]. Summing over the errors, and

[1] which follows from the fact that for a function $f \in \mathcal{S}(\mathbb{R}^n)$, the integral $\hat{f} = \int_{-\infty}^{\infty} f(t')\, e^{-i\omega t'}\, dt' \in \mathcal{S}(\mathbb{R}^n)$

substituting the result in (30), (31), we get

$$\lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-\infty}^{\infty} f(t')e^{-\frac{2\pi i n t'}{T}}\,dt'\; e^{\frac{2\pi i n t}{T}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega + \lim_{T\to\infty} e(T) \quad (32)$$

where e can be bounded from the lines above as

$$|e(T)| \leq \sum_{n\geq 0} \frac{C}{T^2}\frac{1}{1+\left(\frac{n}{T}\right)^2} \quad (33)$$

The terms in (33) are monotonically decreasing and we can thus use the bound

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+\left(\frac{n}{T}\right)^2} \leq \sum_{n=0}^{\infty} \frac{2}{1+\left(\frac{n}{T}\right)^2} \leq \int_0^{\infty} \frac{2}{1+\left(\frac{x}{T}\right)^2}\,dx \leq 2TC' \quad (34)$$

In the last line we use the change of variable $x \leftarrow \frac{x}{T}$ and introduce the absolute constant C' (independent of T). From (33) and (34), we see in particular that $\lim_{T\to\infty} e(T) \leq \lim_{T\to\infty} \frac{C}{T} \to 0$. Substituting this in (32), and going back to (20), we can conclude the derivation with

$$f(t) = \lim_{T\to\infty} g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega \quad (35)$$

Extending the Fourier transform: the notion of distributions

Now that we have introduced the Fourier transform, we might wonder what are the functions on which such a transform can be applied. As an example, what about periodic functions on which we originally defined the Fourier series? and if we can handle periodic functions, can we handle constant functions as well?

Although integrating periodic or constant functions on the whole real line might at first appear as a wrong idea, a more careful look at the integral

$$\int_{-\infty}^{\infty} e^{-i\omega t}\,dt$$

however shows an interesting fact: this integral can read as the sum

$$\int_{-\infty}^{\infty} e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} \cos\omega t\,dt + i\int_{-\infty}^{\infty} \sin\omega t\,dt$$

Moreover, this sum can be decomposed as

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-i\omega t}\,dt &= \int_{-\infty}^{\infty} \cos\omega t\,dt + i\int_{-\infty}^{\infty} \sin\omega t\,dt\\ &= \sum_{k=-\infty}^{\infty} \int_{[(2k+1)\pi/2-\pi/2]/\omega}^{[(2k+1)\pi/2+\pi/2]/\omega} \cos\omega t\,dt\\ &+ i\sum_{k=-\infty}^{\infty} \int_{(k\pi-\pi/2)/\omega}^{(k\pi+\pi/2)/\omega} \sin\omega t\,dt\end{aligned}$$

In particular, from this last decomposition, it is tempting to conclude that, when $\omega \neq 0$, the integral vanishes. When $\omega = 0$ on the other hand, there is no such cancelation and the integral remains unbounded.

Although our transform vanishes almost everywhere, the blow-up that occurs at 0 is unfortunately inappropriate, in particular if we want to keep the good properties of the Fourier transform (e.g. the Fourier inversion theorem). How can we integrate a function that is undefined at one point? One solution could be to consider a transform that would be defined from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n \setminus \{0\})$ but then what about the functions $e^{i\omega_0 t}$ for any ω_0 whose Fourier transforms share the properties of the Fourier transform of 1 yet with a singularity now located at ω_0? each of those would require a separate transform.

The advantage of Schwartz functions is that those functions have very interesting properties, some of which we have already covered above. In particular,

i) If f is in $\mathcal{S}(\mathbb{R}^n)$, then the Fourier transform of f, $\mathcal{F}\{f\}$ is in $\mathcal{S}(\mathbb{R}^n)$ as well.

ii) Those functions satisfy the inversion theorem. if f is in $\mathcal{S}(\mathbb{R}^n)$, then $\mathcal{F}^{-1}\mathcal{F}f = f$

To be able to integrate sines and cosines while preserving essential properties of the Fourier transform, a better solution (than redefining the transform itself) would be to study extensions of the Schwartz class that would preserve those essential properties. Moreover, we would like the extension. to handle the pointwise blow up appearing when taking the Fourier transform of periodic or constant functions.

A first solution to this problem can be found by considering a class of objects known as tempered distributions which will enable us to handle the case of Schwartz functions but also sines, cosines (i.e. classical periodic functions) as well as constant functions.

The class of tempered distributions can be interpreted as the limit of Schwartz functions. As an example of this, consider the family of Gaussians

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad t > 0 \tag{36}$$

Those functions are in the Schwartz class.

Since we saw that the Fourier transform of the function $f(t) = 1$, $\int_{-\infty}^{\infty} e^{i\omega t}\, dt$, could be considered as vanishing everywhere except at $\omega = 0$, it seems that this object could be captured by the limit for $t \to 0$ of the family $g(x,t)$. I.e. if we denote this object as $\delta(x)$, then informally it seems that

$$\delta(x) = \lim_{t\to\infty} g(x,t) \approx \mathcal{F}\{1\}$$

The limit does not define a function though and so defining $\delta(x)$ in this way does not really make sense. What we can however do is compute the integral

$$\int_{-\infty}^{\infty} g(x,t)\, \varphi(x)\, dx$$

Now taking the limit $t \to 0$, we get

$$\lim_{t\to 0} \int_{-\infty}^{\infty} g(x,t)\varphi(x)\, dx = \varphi(0) \tag{37}$$

although the limit of $g(x,t)$ does not define a function and therefore cannot be integrated (i.e. the limit cannot be moved inside the integral in (37)), the limit, when taken with respect to the family of integrals, is itself well defined and corresponds to taking the value of the function $\varphi(x)$ at the mean of an increasingly concentrated Gaussian.

Since our expression in (37) is well defined, it thus seems like a good idea to use it to encode our new mathematical object δ. The price to pay however, is that this δ has to be considered within the integral (i.e as multiplying a function φ). In this setting we can say that applying δ to a function φ returns the value of this function at 0, i.e.

$$\langle \delta, \varphi \rangle = \lim_{t\to 0} \int_{-\infty}^{\infty} g(x,t)\varphi(x)\, dx = \varphi(0) \tag{38}$$

Extending the notion of function, a tempered distribution is thus defined not explicitely but by means of its application (through an integral such as (38)) to a function φ (which will be called test function).

Now let us go back to the Fourier transform. Why would one want to consider the Fourier transform of extremely concentrated Gaussians? Well, such objects have in fact a number of applications throughout mathematics, physics and engineering [6]. Heaviside for example, use the δ in his study of Electromagnetism. This same δ was also used to study how systems respond to sharp/sudden inputs in engineering. Finally Dirac used this object in quantum mechanics.

Now that we have clarified how we planned to use our new class of objects, let us formalize the definition of this class.

Definition 6. *A tempered distribution is a complex valued continuous linear functional on the collection $\mathcal{S}$ of Schwartz functions (called test functions in this setting). The set of tempered distributions will be denoted as $\mathcal{T}$.*

For those who are familiar with the terminology, one can show that $\mathcal{T}$ is the "dual space" of $\mathcal{S}$. By linear functional on the space of Schwartz functions, what we really mean is that we pass test functions to a tempered distribution and it returns a number. As we said earlier, the distribubtion δ in this setting corresponds to evaluating any test function φ at 0 (the fact that δ is a tempered distribution follows from the fact that the space of tempered distributions is closed, hence the limit of a sequence of functions from $\mathcal{T}$ belongs to $\mathcal{T}$).

Let us now try to understand the relation between tempered distributions and the Fourier transform. To get a sense of how general operators on functions can be extended to tempered distributions, a good first step could be to study the distributions derived from classical Schwartz functions (for which the Fourier transform is well defined) and consider how to generalize the Fourier transform from there. For any traditional Schwartz function, we can define a tempered distribution as

$$T_f(\varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx$$

In other words, every Schwartz function induces a tempered distribution. We should also keep in mind that our objective is to end up with a Fourier transform on the space of distribution for which most of the properties of the Fourier transform that we defined on Schwartz functions still hold. In particular, we would like the Fourier transform of a distribution to be another distribution. And if the distribution is defined from an associated Schwartz function, then it seems natural to define the Fourier transform of this distribution as the distribution associated to the Fourier transform of the Schwartz function. I.e for any Schwartz function f, our definition of $\mathcal{F}T_f$ should obey

$$\mathcal{F}\{T_f\} = \int_{-\infty}^{\infty} \mathcal{F}\{f\}\varphi(x)\, dx = \langle \mathcal{F}\{f\}, \varphi\rangle = T_{\mathcal{F}\{f\}} \tag{39}$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-2\pi i t x} f(t)\, dt \right) \varphi(x)\, dx \tag{40}$$

Provided that the function $e^{-2\pi i t x} f(t)\ \varphi(x)$ is absolutely integrable, we can interchange the integrals (recall that both f and φ are Schwartz functions). From this, we get

$$\begin{aligned}
\langle \mathcal{F}f, \varphi\rangle &= \int_{-\infty}^{\infty} \mathcal{F}\{f\}\ \varphi(x)\, dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i x t}\ f(t)\, dt\, \varphi(x)\, dx \\
&= \int_{-\infty}^{\infty} f(t)\ \left(\int_{-\infty}^{\infty} e^{-2\pi i x t}\varphi(x)\, dx \right) dt \\
&= \langle f, \mathcal{F}\varphi\rangle
\end{aligned}$$

We therefore see that, in the case of Schwartz functions, if we define the Fourier transform of the distribution to be the distribution associated to the Fourier transform of the Schwartz function, the Fourier transform of the distribution is the linear operator which, when given a test function φ returns the integral $\int_{-\infty}^{\infty} f(x)\ \mathcal{F}\{\varphi\}\ dx$

Now the good news is that this new definition for the Fourier transform is in fact more powerful than the original one as it does not require the Fourier transform of f to be a Schwartz function (as a matter of fact it does not even require the classical Fourier transform of f to be defined). Instead, the Fourier transform of tempered distributions is now defined by saying that for any tempered distribution T_f, the Fourier transform returns a linear operator that takes as input a (Schwartz) test function φ and returns the integral

$$(\mathcal{F}T_f)(\varphi) = \langle f, \mathcal{F}\varphi\rangle = \int_{-\infty}^{\infty} f(t)\, dt \int_{-\infty}^{\infty} e^{-2\pi i x t}\ \varphi(x)\, dx$$

As a result, we can now handle the Fourier transform of 1 (in the distribution sense),

$$\begin{aligned}\mathcal{F}T_1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi ixt}\,\varphi(x)\,dx \\ &= \int_{-\infty}^{\infty} e^{2\pi it0}\int_{-\infty}^{\infty} e^{-2\pi ixt}\varphi(x)\,dx\,dt \\ &= \mathcal{F}\left\{\mathcal{F}\varphi\right\}(0) = \varphi(0)\end{aligned}$$

as the (tempered) distribution that, when applied to a test function φ returns the value of the this test function at 0.

1 The discrete Fourier transform

The Discrete Fourier Transform (DFT) is one of the most important tools in digital signal processing. Because ot its ability to store a signal and compute the Fourier transform of this signal with a finite amount of operations and data, it can be implemented in computers by numerical algorithms or even dedicated hardware.

Consider a 2π-periodic continuous signal $X(t)$. Assume that $X(t)$ can be represented by an absolutely convergent trigonometric Fourier series

$$X(t) = \sum_{m=-\infty}^{\infty} c_m e^{imt}, \quad t \in [-\pi, \pi] \tag{41}$$

where c_m are the Fourier coefficients of $X(t)$. Let N be an even positive integer and let us define t_k as

$$t_k = \frac{2\pi k}{N}, \quad k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$$

Sampling $X(t)$ at the values t_k, we then get

$$X(t_k) = \sum_{m=-\infty}^{\infty} c_m e^{i\frac{2\pi ikm}{N}}$$

Since $e^{2\pi ik\ell} = 1$ for any integers k, ℓ, we can rewrite the series (41) by using a

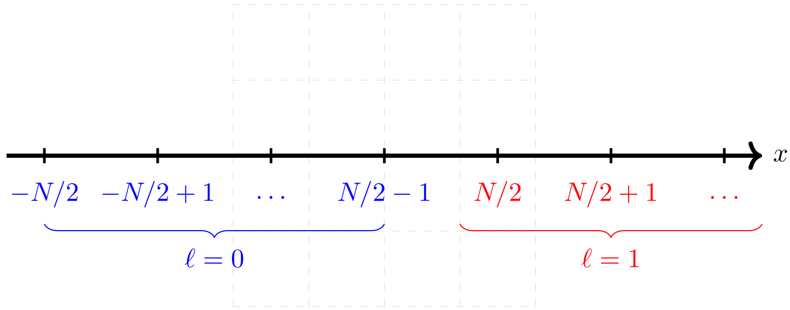

Figure 1: Decomposition used for the DFT (44)

decomposition illustrated in Fig. 1

$$X(t_k) = \sum_{m=-\infty}^{\infty} c_m e^{\frac{2\pi i k}{N}(m-\ell N)} \tag{42}$$

$$= \sum_{\ell=-\infty}^{\infty} \sum_{n=-N/2}^{N/2-1} c_{n+\ell N} e^{\frac{2\pi i k((n+\ell N)-\ell N)}{N}} \tag{43}$$

$$= \sum_{\ell=-\infty}^{\infty} \sum_{n=-N/2}^{N/2-1} c_{n+\ell N} e^{\frac{2\pi i k n}{N}} \tag{44}$$

$$= \sum_{n=-N/2}^{N/2} e^{\frac{2\pi i k}{N} n} \sum_{\ell=-\infty}^{\infty} c_{n+\ell N} \tag{45}$$

$$= \sum_{n=-N/2}^{N/2-1} e^{\frac{2\pi i k n}{N}} \hat{x}_n \tag{46}$$

where we have defined the $\hat{x}_n$ as

$$\hat{x}_n = \sum_{\ell=-\infty}^{\infty} c_{n+\ell N} \tag{47}$$

The formula (46) can be viewed as an inverse Fourier transform and appears quite naturally in the discretization of periodic signals.

The recovery of the coefficients $\hat{x}_n$ from the samples $X(t_k)$ can in fact be viewed as an interpolation problem

$$X(t_k) = \sum_{n=-N/2}^{N/2-1} \hat{x}_n e^{\frac{2\pi i k n}{N}}, \qquad t_k = \frac{2\pi k}{N}, \quad k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1 \tag{48}$$

on the orthogonal basis functions $e^{\frac{2\pi i k n}{N}}$. I.e.

$$\sum_{k=-N/2}^{N/2-1} e^{i(m-n)\frac{2\pi k}{N}} = \begin{cases} 0 & m - n \neq 0, \pm N, \pm 2N, \ldots \\ N & m - n = 0, \pm N, \pm 2N, \ldots \end{cases} \tag{49}$$

This problem can be solved exactly as we have (using (48))

$$\begin{aligned}\sum_{k=-N/2}^{N/2-1} X_k e^{-\frac{2\pi ikn}{N}} &= \sum_{k=-N/2}^{N/2-1}\sum_{m=-N/2}^{N/2-1} \hat{x}_m e^{\frac{2\pi ikm}{N}} e^{-\frac{2\pi ikn}{N}} \\ &= \sum_{m=-N/2}^{N/2-1}\sum_{k=-N/2}^{N/2-1} \hat{x}_m e^{\frac{2\pi ik}{N}(m-n)} \\ &= N\hat{x}_n\end{aligned}$$

from which we get

$$\hat{x}_n = \frac{1}{N}\sum_{k=-N/2}^{N/2-1} X_k e^{-\frac{2\pi ikn}{N}} \tag{50}$$

A discrete equivalent of the Fourier transform. The sequences $\{X_k\}_{k=-N/2}^{N/2-1}$ and $\{\hat{x}_n\}_{n=-N/2}^{N/2-1}$ are known as discrete Fourier transform pairs. This idea is summarized by Definition 7 below.

Definition 7. *The sequence $\{\hat{x}_n\}_{n=-N/2}^{N/2-1}$ of complex numbers is called the Discrete Fourier transform (DFT) of the sequence $\{X_k\}_{k=-N/2}^{N/2-1}$ if for each $n = -N/2, \ldots, N/2-1$, we have*

$$\hat{x}_n = \frac{1}{N}\sum_{k=-N/2}^{N/2-1} X_k e^{-\frac{2\pi ikn}{N}}$$

Similarly, the sequence $\{X_k\}_{k=-N/2}^{N/2-1}$ of complex numbers is said to be the inverse Discrete Fourier transform (IDFT) of the sequence $\{x_n\}_{n=-N/2}^{N/2-1}$ if for each $k = -N/2, \ldots, N/2-1$, we have

$$X_k = \sum_{n=-N/2}^{N/2-1} \hat{x}_n e^{\frac{2\pi ikn}{N}} \tag{51}$$

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