MATH-UA 9263 - Partial Differential Equations
Recitation 9: Wave equation (part II)

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Question 1 *The chord of a guitar of length L is plucked at its middle point and then released. Write the mathematical model which governs the vibrations and solve it. Compute the energy $E(t)$.*

Question 2 *Solve the problem*

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1, t > 0 \\ u(x,0) = u_t(x,0) = 0 & 0 \leq x \leq 1 \\ u_x(0,t) = 1, u(1,t) = 0 & t \geq 0 \end{cases}$$

Question 3 *Solve the problem*

$$\begin{cases} u_{tt} - u_{xx} = g(t)\sin x & 0 < x < \pi, t > 0 \\ u(x,0) = u_t(x,0) = 0 & 0 \leq x \leq \pi \\ u(0,t) = u(\pi,t) = 0 & t \geq 0 \end{cases}$$

Question 4 (Equipartition of energy) . *Let $u = u(x,t)$ be the solution of the global Cauchy problem for the equation $u_{tt} - cu_{xx} = 0$ with initial data $u(x,0) = g(x)$, $u_t(x,0) = h(x)$. Assume that g and h are smooth functions with compact support contained in the interval (a,b). Show that there exists T such that, for $t \geq T$,*

$$E_{cin}(t) = E_{pot}(t)$$

Question 5 *Consider waves in a resistant medium which satisfies the following PDE*

$$\begin{aligned} u_{tt} &= c^2 u_{xx} - r u_t & 0 < x < \ell \\ u(0,t) &= u(\ell,t) = 0, & \forall t > 0 \\ u(x,0) &= \phi(x) & 0 < x < \ell \\ u_t(x,0) &= \psi(x) & 0 < x < \ell \end{aligned}$$

where r is a constant. Write down a series expansion for the following cases

1. $0 < r < \frac{2\pi c}{\ell}$

2. $\frac{2\pi c}{\ell} < r < \frac{4\pi c}{\ell}$

You may assume that the initial conditions can be represented using an appropriate Fourier series.

Question 6 *Solve the following problems:*

1. $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$
2. $u_{tt} = c^2 u_{xx}$, $u(x,0) = \log(1+x^2)$, $u_t(x,0) = 4+x$

Question 7 *The midpoint of a piano string of tension T, density ρ and length ℓ is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $\ell/4$ from one end (Assume that $a < \ell/4$). How long does it take for the disturbance to reach the flea?*

Question 8 *Let $u(x,0) = \phi(x) = 0$ (initial position) and $u_t(x,0) = \psi(x) = 1$ (initial velocity) for $|x| \geq a$. Sketch the string profile (u versus x) ar each of the successive instants $t = a/2c$, a/c, $3a/2c$, $2a/c$ and $5a/c$ where c is the wave speed*

$$u_{tt} = c^2 u_{xx}$$

[Hint: Calculate

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds = \frac{1}{2c}\,[\textit{length of}(x-ct, x+ct) \cap (-a,a)]$$

Then $u(x, a/2c) = (1/2c)$ (length of $(x-a/2, x+a/2) \cap (-a,a)$). This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.]

Question 9 *In question 8, find the greatest displacement, $\max_x u(x,t)$ as a function of t.*

Question 10 *A spherical wave is a solution of the three dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation then takes the form*

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right), \quad \textit{spherical wave equation}$$

(a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$

(b) Solve for v using

$$u(x,t) = f(x+ct) + g(x-ct)$$

and thereby solve the spherical wave equation.

(c) Use d'Alembert's decomposition

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

where $u(x,0) = \phi(x), u_t(x,0)\psi(x)$ *to solve the wave equation with initial conditions* $\phi(r)$ *and* $\psi(r)$. *Taking* $\phi(r)$ *and* $\psi(r)$ *to be even functions of* r

Question 11 *Solve the problem*

$$\begin{cases} u_{xx} - 3u_{xt} - 4u_{tt} = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{cases}$$

[Hint: Factor the operator as we did for the wave equation]

References

[1] Kenneth S. Miller, *Partial Differential Equations in Engineering Problems*, Dover Publications inc. 2020.

[2] Walter A. Strauss, *Partial Differential Equations An Introduction*, John Wiley and Sons Ltd, 2008

[3] Sandro Salsa, *Partial Differential Equations in Action, From Modelling to Theory*, Springer, 2016.

[4] Lawrence C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, 2010.