

# Partial Differential Equations - MATH-UA 9263

## Final II

Augustin Cosse

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**Total:** 28 points

**Total time:** 1h30

**General instructions:** The exam consists of 3 questions (each question consisting itself of 2 or 3 subquestions). Once you are done, make sure to write your name on each page, then take a picture of all your answers and send the file by email to [acosse@nyu.edu](mailto:acosse@nyu.edu). In case you have any question, you can ask those through the chat. Answer as many questions as you can starting with those you feel more confident with.

### Question 1 (10pts)

1. [6pts] We want to solve the following problem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + cu = 0 & t > 0, 0 \leq x \leq L \\ \frac{\partial u}{\partial x}(t, 0) = 0 & t > 0 \\ u(t, L) = 0 & t > 0 \\ u(0, x) = f(x) & 0 \leq x \leq L \end{cases} \quad (1)$$

where  $c \in \mathbb{R}$  and  $L > 0$  are fixed constants and  $f(x)$  is piecewise  $C^1$  on  $[0, L]$ .

(a) Find  $\mu \in \mathbb{R}$  such that if we apply the change of variable  $v(x, t) = e^{\mu t} u(x, t)$ , we can reduce (1) to the heat equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \quad (2)$$

Write down the corresponding system (including initial and boundary conditions) on  $v$

(b) We want to find the solutions of (2) of the form  $v(x, t) = T(t)X(x)$ . Find the general form of  $T(t)$  and  $X(x)$ .

(c) Show that if we look for  $T(t)$  and  $X(x)$  that satisfy the boundary conditions, we get an infinite number of solutions whose expressions are of the form  $v_k(x, t) = T_k(t)X_k(x)$  where  $k = 0, 1, 2, \dots$ . Give the expression of the the functions  $T_k(t)$  and  $X_k(x)$ .

(d) Find the solution  $v$  of the form

$$v(x, t) = \sum_{k=1}^{\infty} T_k(t)X_k(x) \quad (3)$$

Then derive the solution  $u$  of (1).

(e) We now assume  $f(x) = \cos(\frac{\pi x}{2L})$ , find the expression for  $u(x, t)$ .

2. [4pts] Solve the following general Cauchy problem (Hint: factor the operator as we did to derive d'Alembert's formula)

$$\begin{cases} u_{xx} - 3u_{xt} - 4u_{tt} = 0 \\ u(x, 0) = x^2, u_t(x, 0) = e^x \end{cases} \quad (4)$$

**Question 2 (10pts)**

1. [5pts] Solve the following initial value problem (Hint: consider  $x + y$  and  $x - y$ )

$$\begin{cases} y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = 0 \\ u(x, 0) = f(x) \end{cases}$$

2. [5pts] We consider the Riemann problem

$$u_t + f(u)_x = 0 \quad \text{in } [0, \infty) \times \mathbb{R}, \quad u(0, x) = u_0(x)$$

with the flux function  $f(u) = u^2(1 - u)^2$ . Further let

$$u_0(x) = \begin{cases} 2 & \text{if } x < 0, \\ 0 & \text{if } x > 0 \end{cases}$$

Construct a weak solution which satisfies the Rankine-Hugoniot condition.

**Question 3 (10pts)**

1. Classify each of the following PDEs as hyperbolic, parabolic or elliptic [3pts].

(a)  $u_{xx} - 5u_{xy} = 0$

(b)  $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$

(c)  $4u_{xx} + 6u_{xy} + 9u_{yy} = 0$

2. Consider  $Q = \{(x, t) \mid 0 < x < L, t > 0\}$  and  $\bar{Q}$  the closure of  $Q$ . Assume that  $u$  and  $v$  are in  $C(\bar{Q}) \cap C^2(Q)$  and respectively solve  $\partial_t u - \partial_x^2 u = f(x, t)$  and  $\partial_t v - \partial_x^2 v = g(x, t)$  on  $Q$ . Furthermore, assume that  $f < g$  on  $Q$  and  $u \leq v$  for  $t = 0$ , for  $x = 0$  and for  $x = L$ . Show that  $u \leq v$  on  $Q$  [Hint: use the proof of the weak maximum principle] [5pts]

3. Define the notion of harmonic function [2pts]