

Example 1

$$\begin{cases} \frac{\partial u}{\partial t} + 2u \frac{\partial u}{\partial x} = 0 \\ u(x, 0) = \begin{cases} 3 & x < 0 \\ 4 & x \geq 0 \end{cases} \end{cases}$$

$$\Gamma(s) = (s, 0) \quad u(\Gamma(s)) = \begin{cases} 3 & s < 0 \\ 4 & s > 0 \end{cases} \quad \phi(s) = \begin{cases} 3 & s < 0 \\ 4 & s > 0 \end{cases}$$

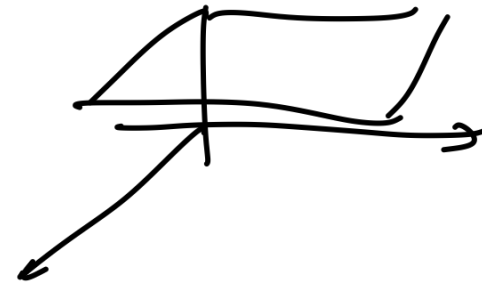
The characteristics obey the relations

$$\frac{dx}{d\xi} = 2u \quad \frac{dt}{d\xi} = 1 \quad \frac{dz}{d\xi} = 0$$

$$I(\xi) \quad x(0) = s \quad t(0) = 0 \quad z(0) = \phi(s) \rightarrow z(\xi) = \phi(s)$$

$$x(s, \xi) = 2\phi(s)\xi + s$$

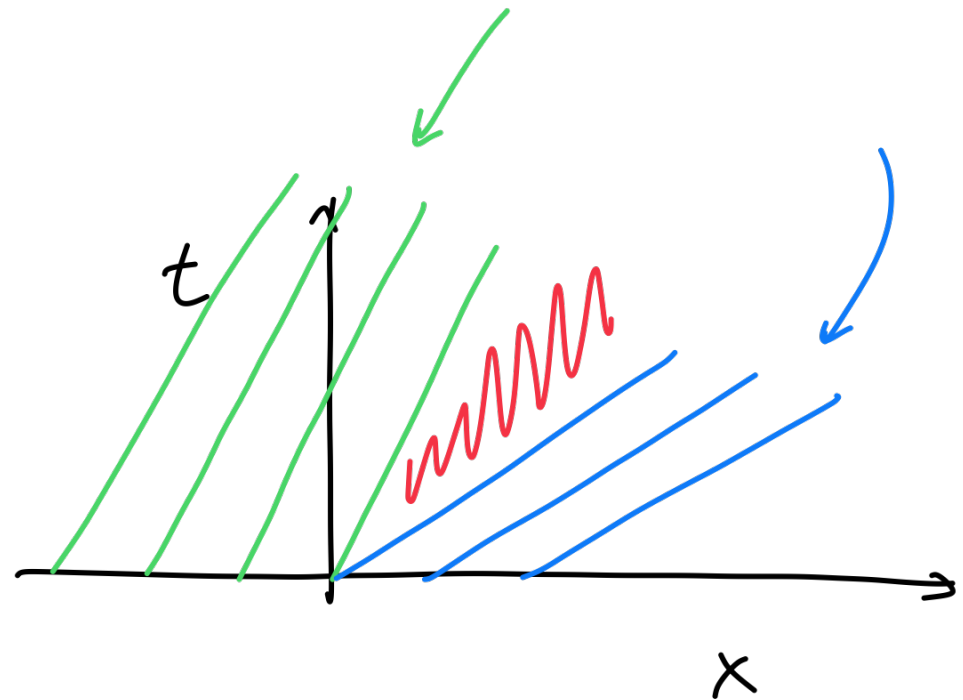
$$t(s, \xi) = \xi$$



$$x = 2\phi(s)t + s$$

$$x(t) = \begin{cases} 6t + s & s < 0 \\ 8t + s & s \geq 0 \end{cases} \quad *$$

$$x(t) = 6t + s$$



using (*) we get

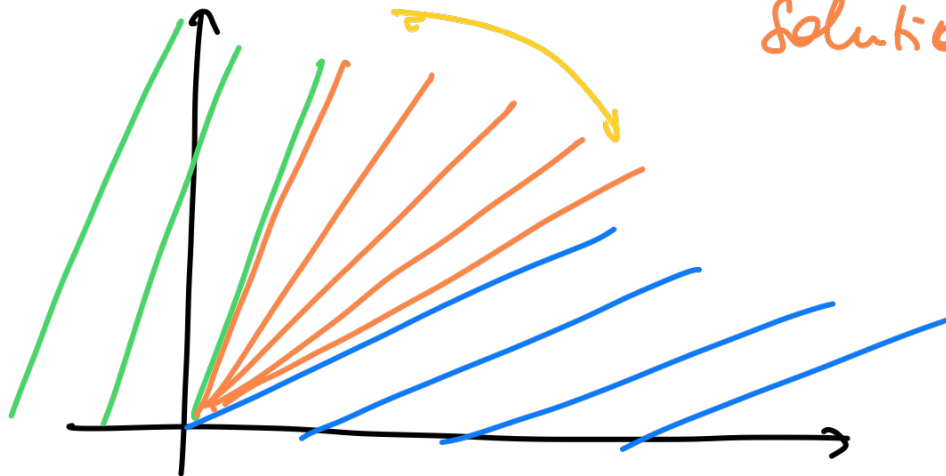
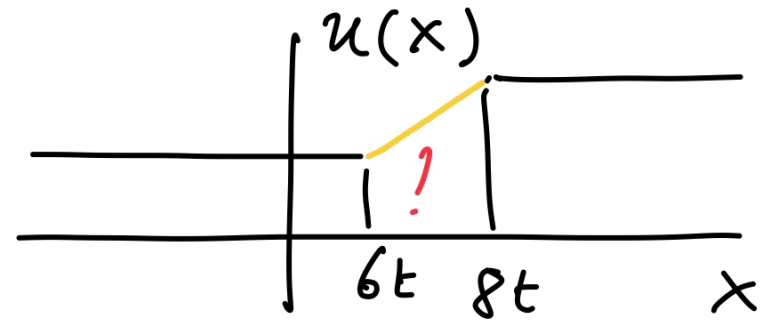
$$s = x - 6t \quad \text{when } s = \overbrace{x - 2\phi(s)t} < 0$$

$$s = x - 8t \quad \text{when } s = \underbrace{x - 2\phi(s)t} \geq 0$$

Substituting this in the expression for u , we get

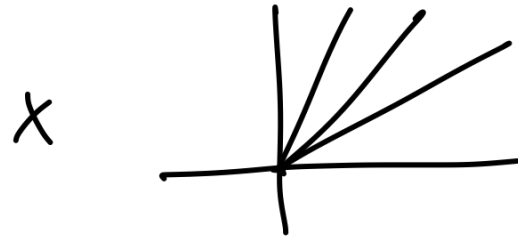
$$u(x, t) = \begin{cases} 3 & \text{if } s < 0 \\ 4 & \text{if } s > 0 \end{cases} = \begin{cases} 3 & \text{if } \overbrace{x - 6t} < 0 \\ 4 & \text{if } x - 8t \geq 0 \end{cases}$$

$$= \begin{cases} 3 & x < 6t \\ 4 & x \geq 8t \end{cases}$$



Solution: introduce fan-like characteristics

$$x = \begin{cases} 6t + s & s < 0 \\ \rho t & 6 < \rho < 8 \\ 8t + s & s > 0 \end{cases}$$



the fan-like characteristics then turn the solution to

$$u(x, t) = \begin{cases} 3 & \text{if } x \leq 6t \\ \rho/2 & \text{if } x = \rho t \\ 4 & \text{if } x > 8t \end{cases} \quad \rho \in [6, 8]$$

