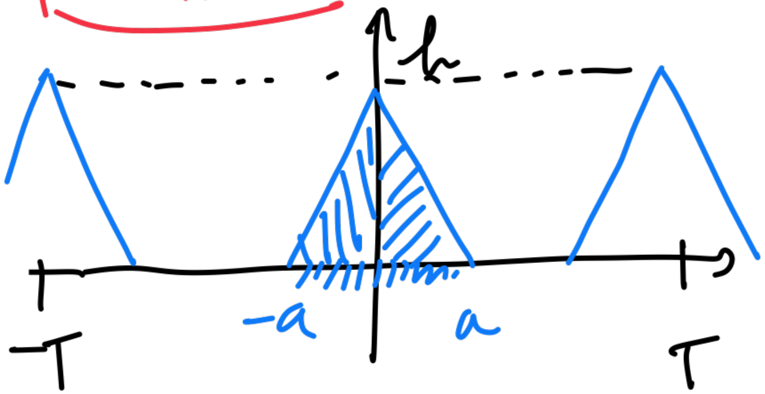
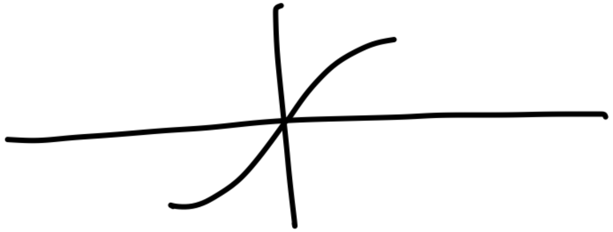
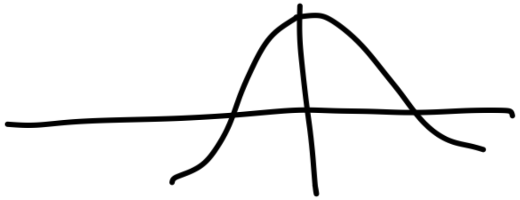
# Question 1




$$f(x) = a_0 + \sum_{k=1}^{\infty} a_n \sin \frac{2\pi n x}{T} + \sum_{k=1}^{\infty} b_n \cos \frac{2\pi n x}{T}$$



$$f(x) = a_0 + \sum_{k=1}^{\infty} b_n \cos \frac{2\pi n x}{T}$$



$$\int_{-T/2}^{T/2} \cos \frac{2\pi n x}{T} \cos \frac{2\pi m x}{T} \, dx$$

$$\cos \frac{2\pi n x}{T} \cos \frac{2\pi m x}{T} = \frac{\cos\left(\frac{2\pi}{T}(m+n)x\right) + \cos\left(\frac{2\pi}{T}(m-n)x\right)}{2}$$

$$\int_{-T/2}^{T/2} \frac{\cos\left(\frac{2\pi}{T}(m+n)x\right) + \cos\left(\frac{2\pi}{T}\cancel{(m-n)}x\right)}{2}\,dx$$

0

$$= 0 \qquad \text{when } m \neq n \neq 0$$

$$= \int_{-T/2}^{T/2} \frac{1 + \cos\left(\frac{2\pi}{T}(m+n)x\right)}{2}\,dx = \frac{T}{2} \qquad \text{when } m = n \neq 0$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \cos \frac{n 2\pi x}{T}$$

0 $\quad \sin\left(\frac{k \cancel{2}\pi}{\cancel{T}} \frac{\cancel{T}}{\cancel{4}}\right)$

$$\int_{-T/2}^{T/2} f(x) \cos\left(\frac{k 2\pi x}{T}\right) dx = \int_{-T/2}^{T/2} a_0 \cancel{\cos\left(\frac{k 2\pi x}{T}\right)}\, dx$$

$$+ \sum_{n=1}^{\infty} \int_{-T/2}^{T/2} b_n \cos\left(\frac{2\pi n}{T}x\right) \cos\left(\frac{2\pi k}{T}x\right) dx$$

$$\int_{-T/2}^{T/2} f(x)\cos\left(k\frac{2\pi x}{T}\right)dx = b_k\frac{T}{2}$$

$$\int_a^b f g' = \left| f\cdot g\right|_a^b - \int_a^b f' g\,dx$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\left(\frac{2\pi k x}{T}\right)dx$$

$$= \frac{2}{T}\underbrace{\int_{-a}^{0}\overbrace{(x+a)\frac{h}{a}}^{f}\,\overbrace{\cos\left(\frac{2\pi k x}{T}\right)}^{g'}} + \frac{2}{T}\underbrace{\int_0^a (x-a)\left(\frac{-h}{a}\right)\cos\left(\frac{2\pi k x}{T}\right)}$$

$$= \frac{2}{T}\left|(x+a)\frac{h}{a}\,\frac{T}{2\pi k}\sin\left(\frac{2\pi k}{T}x\right)\right|_{-a}^{0}$$

$$-\frac{2}{T}\int_{-a}^{0}\frac{T}{2\pi k}\sin\left(\frac{2\pi k}{T}x\right)dx\,\frac{h}{a}$$

$$= 0 + \frac{2}{T}\left(\frac{T}{2\pi k}\right)^2 \left| \cos\left(\frac{2\pi k}{T}x\right) \right|_{-a}^{0}$$

$$= \frac{2}{T}\left(\frac{T}{2\pi k}\right)^2 \left(1 - \cos\left(\frac{2\pi k a}{T}\right)\right) \frac{h}{a} \Bigg)$$

$$\frac{2}{T}\int_0^a \overbrace{(x-a)}^{f}\left(-\frac{h}{a}\right) \overbrace{\cos\left(\frac{2\pi k}{T}x\right)}^{g'} dx$$

$$= \frac{2}{T}\left| \underbrace{(x-a)\left(-\frac{h}{a}\right)}_{0} \cancel{\frac{T}{2\pi k}\sin\left(\frac{2\pi k}{T}x\right)} \right|_0^a - \frac{2}{T}\int_0^a -\frac{h}{a}\,\frac{T}{2\pi k}\sin\left(\frac{2\pi k}{T}x\right)dx$$

$$-\frac{h}{a}\left(\frac{T}{2\pi k}\right)^2 \left| \cos\left(\frac{2\pi k}{T}x\right) \right|_0^a \frac{2}{T}$$

$$= \frac{2}{T}\frac{h}{a}\left(\frac{T}{2\pi k}\right)^2 \left(1 - \cos\left(\frac{2\pi k}{T}a\right)\right) \Bigg)$$

$$b_k = \frac{4}{T}\frac{h}{a}\left(\frac{T}{2\pi k}\right)^2\left(1-\cos\left(\frac{2\pi k}{T}a\right)\right) \text{ when } k>0$$

when $k=0$

$$f(x) = \underbrace{a_0} + \sum_{n=1}^{\infty} b_n \cos\frac{2\pi n}{T}x$$

$$\int_{-T/2}^{T/2} f(x)\,dx = \underbrace{\int_{-T/2}^{T/2} a_0\,dx} + \int_{-T/2}^{T/2}\cancel{\sum_n b_n \cos\frac{2\pi n x}{T}\,dx}^{\,0}$$

$$\int_{-a}^{a} \underbrace{f(x)\,dx}_{2\cdot\frac{ah}{2}} = T a_0$$

$$\Longrightarrow a_0 = \frac{1}{T}\,ah$$

$$\int_{-T}^{T} e^{-i\frac{\pi m x}{T}} f(x) = \int_{-T}^{T} \sum_{n=-\infty}^{\infty} \underbrace{c_n e^{i\frac{\pi}{T}nx} \; e^{-i\frac{\pi m}{T}x}}_{\to\ \underline{c_m 2T}}$$

Complex Fourier series

$$\int_{-T}^{T} e^{i\frac{\pi n x}{T}} \, e^{-i\frac{\pi m x}{T}} \, dx = 0 \quad \text{when } n \neq m$$

$$= \frac{T}{i\pi(n-m)} \left| \underbrace{e^{i\frac{\pi(n-m)x}{T}}}_{} \right|_{-T}^{T}$$

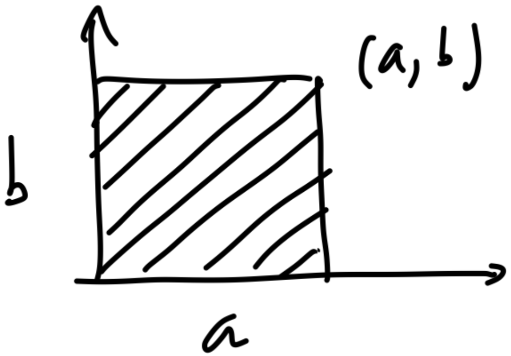
$$= \frac{T}{i\pi(n-m)} \left[ \underbrace{e^{i\pi(n-m)}}_{} - \underbrace{e^{-i\pi(n-m)}}_{} \right] = 0$$

$$\cancel{\cos(\pi(n-m))} + i \underbrace{\cancel{\sin(\pi(n-m))}}_{0} \; - \; \cancel{\cos(\pi(n-m))} + i \underbrace{\cancel{\sin(\pi(n-m))}}_{0}$$

$$\int_{-T}^{T} \underbrace{e^{i\frac{\pi(n)x}{T}} \; e^{-i\frac{\pi(n)x}{T}}}_{1} \, dx = 2T$$

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(x) \, e^{-\frac{i 2\pi n x}{T}}$$

Q7




$$\begin{cases} u(0,y) = 0 \\ u(a,y) = 0 \\ u(x,b) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$\cancel{\frac{\partial u}{\partial t}} = D \Delta u \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x,y) = v(x)\,w(y) \qquad (*)$$

let us substitute (*) inside our steady state heat equation

$$v''(x)\,w(y) + v(x)\,w''(y) = 0$$

let us separate the variables

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda$$ eigen problem

$\lambda = 0 \Rightarrow v''(x) = 0 \Rightarrow v(x) = Ax + B$

using BC 1 $\rightarrow v(0)w(y) = 0 \rightarrow v(0) = 0 \rightarrow B = 0$

BC 2 $\rightarrow v(a)w(y) = 0 \rightarrow v(a) = 0 \rightarrow Aa = 0$

$\rightarrow$ trivial $\qquad A = 0$

$\lambda > 0 \qquad \frac{v''(x)}{v(x)} = \lambda = \mu^2 \rightarrow v(x) = Ae^{\mu x} + Be^{-\mu x}$

BC 1 $\rightarrow v(0) = 0 \rightarrow A + B = 0$

BC 2 $\rightarrow v(a) = 0 \rightarrow A(e^{\mu a} - e^{-\mu a}) = 0$

$A = 0$

$\lambda < 0$

$$\frac{v''(x)}{v(x)} = \lambda = -\mu^2 \longrightarrow v(x) = A e^{i\mu x} + B e^{-i\mu x}$$

$$BC1 \to v(0) \to A = -B$$

$$BC2 \quad v(a) \to A\left(e^{i\mu a} - e^{-i\mu a}\right) = 0$$

$$2Ai \sin(\mu a) = 0$$

$$\mu a = k\pi \qquad \mu = \frac{k\pi}{a} \qquad k = 1, \dots$$

$$v_k(x) = A_k \sin\left(\frac{k\pi}{a} x\right)$$

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$$\left(\frac{w''(y)}{w(y)}\right) = -\lambda = \mu^2 = \left(\frac{k\pi}{a}\right)^2$$

$$W(y) = A e^{\left(\frac{k\pi}{a}\right)^2 y} + B e^{-\left(\frac{k\pi}{a}\right)^2 y}$$

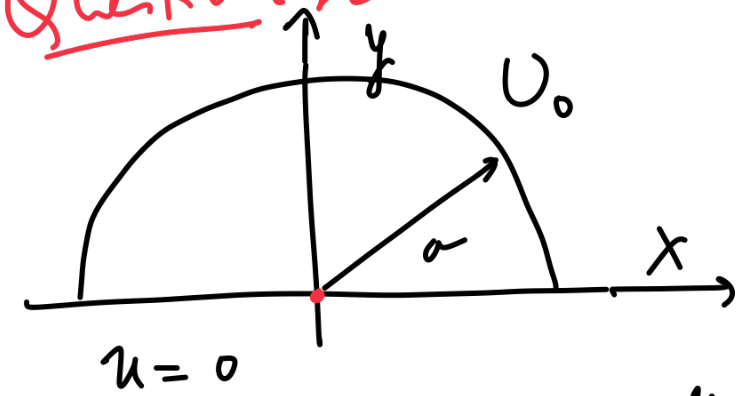
$$W(b) = 0 \qquad A e^{\left(\frac{k\pi}{a}\right)^2 b} + B e^{-\left(\frac{k\pi}{a}\right)^2 b} = 0$$

$$= A e^{2\left(\frac{k\pi}{a}\right)^2 b} = -B$$

$$W_k(y) = A\left( e^{\left(\frac{k\pi}{a}\right)^2 y} - e^{2\left(\frac{k\pi}{a}\right)^2 b} e^{-\left(\frac{k\pi}{a}\right)^2 y} \right)$$

$$U_k(x,y) = v_k(x)\, W_k(y)$$

Question 10




$$\begin{cases} u(a,\theta) = U_0 \\ u(r,0) = u(r,\pi) = 0 \end{cases}$$

$$\cancel{u_t} - D\Delta u = 0$$

$\Delta u = 0$ Laplacian in cylindrical coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad (*)$$

$$u(r,\theta) = v(r)w(\theta)$$

Substitute $u(r,\theta)$ into $(*)$

$$v''(r)w(\theta) + \frac{1}{r}v'(r)w(\theta) + \frac{1}{r^2}v(r)w''(\theta) = 0$$

$$\frac{v''(r) + \frac{1}{r} v'(r)}{\frac{1}{r^2} \quad v(r)} = -\frac{w''(\theta)}{w(\theta)} \quad = \lambda = \mu^2$$

let us start with $v(r)$

$$v''(r) + \frac{1}{r} v'(r) = \lambda \frac{1}{r^2} v(r) \quad (\ast\ast)$$

$$r^2 v''(r) + r v'(r) - \lambda v(r) = 0 \quad \text{( Equidimensional / Euler / Cauchy )}$$

Substitute $v(r) = r^p$

$$\left( r^2 \frac{d^2}{dr^2} + r \frac{d}{dr} - \lambda \quad \right) \leftarrow$$

For the associated operator any function of the form $v(r) = r^p$ reproduces itself.

substitute $v(r) = r^p$ in (**) we get

$$p(p-1)r^p + pr^p - \lambda r^p = 0$$

$$(\underbrace{p(p-1) + p} - \lambda) r^p = 0$$

$$(p^2 - \lambda) r^p = 0 \qquad p = \pm\sqrt{\lambda} = \pm\mu$$

exceptional when $\lambda = 0$

(**) reduces $\quad r^2 v''(r) + r v'(r) = 0$

$$r v''(r) + v'(r)$$

$$(r v'(r))' = 0$$

$$r v'(r) = \text{cst} \Rightarrow v'(r) = \frac{c}{r}$$

$$v(r) = C\log r + B$$

Recap $\quad \lambda = 0 \rightarrow v(r) = C\log r + B$

$\qquad\quad \lambda \neq 0 \rightarrow v(r) = C r^{\mu} + B r^{-\mu}$

Another way to convince ourselves that $r^{p}$ is the solution of $(**)$, take $r = e^{s}$ $\qquad s = \log(r)$

substitute in ODE $**$

$$r^2 \frac{d^2 v}{dr^2} + r\frac{dv}{dr} - \lambda v(r)$$

$$\frac{dv}{dr} = \frac{dv}{ds}\cdot\frac{ds}{dr} = \frac{dv}{ds}\cdot\frac{1}{r}$$

$$\frac{d^2 v}{dr^2} = \frac{d}{dr}\left(\frac{dv}{ds}\cdot\frac{ds}{dr}\right) = \frac{d}{dr}\left(\frac{dv}{ds}\cdot\frac{1}{r}\right)$$

$$= \frac{d^2 v}{ds^2}\cdot\left(\frac{1}{r}\right)^2 + \frac{dv}{ds}\cdot -\frac{1}{r^2}$$

$$r^2\left(\frac{d^2 v}{ds^2}\left(\frac{1}{r^2}\right) + \underbrace{\frac{dv}{ds}\cdot -\frac{1}{r^2}}\right) + \underbrace{r\,\frac{dv}{ds}\cdot}\frac{1}{r} - \lambda v(r) = 0$$

$$\frac{d^2 v}{ds^2} - \lambda v(s) \longrightarrow \quad v(s) = Ae^{\mu s} + Be^{-\mu s}$$

$$r = e^s \qquad = Ar^{\mu} + Br^{-\mu}$$

Solution in $\theta$

$$\frac{w''(\theta)}{w(\theta)} = -\lambda \longrightarrow w(\theta) = Ae^{i\mu\theta} + Be^{-i\mu\theta}$$

at $\theta = 0$ and $\theta = \pi$, using the Boundary conditions, we get

$$W(0) = W(\pi) = 0$$

$$\rightarrow \quad A + B = 0$$

$$A e^{i\mu\pi} - A e^{-i\mu\pi} = 0$$

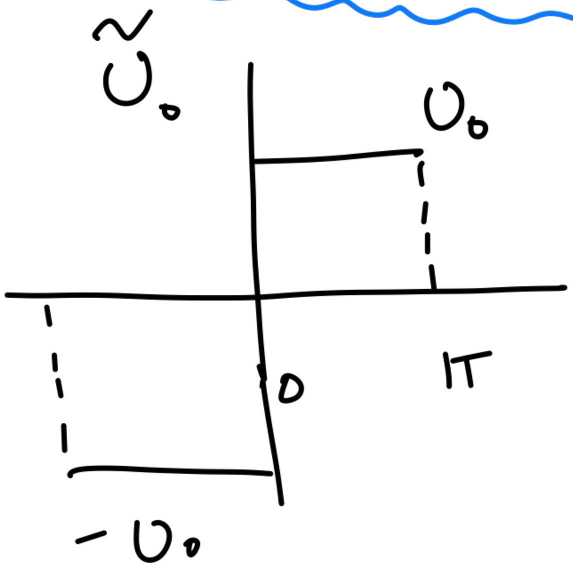
$$A \sin \mu\pi = 0$$

$$\mu = k \quad \text{for } k = 1, \dots$$

$$U_k(r,\theta) = A_k r^{\mu k} \sin k\theta$$

$$U(r,\theta) = \sum_{k=1}^{\infty} A_k r^k \sin k\theta$$

$$U(a,\theta) = \sum_{k=1}^{\infty} A_k a^k \sin k\theta = U_0$$




$$\int_{-\pi}^{\pi} \tilde{U}_0 \sin n\theta \, d\theta$$

$$= \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} A_k a^k \sin k\theta \sin n\theta \, d\theta$$

$$\pi A_n a^n = 2 \left| -\frac{\cos n\theta}{n} \right|_0^{\pi}$$

$$= \frac{4}{n}$$

$$A_n a^n = \frac{4}{n\pi} \qquad A_n = \frac{4}{n\pi a^n}$$

$$U(r,\theta) = \sum_{k=1}^{\infty} \frac{4}{n\pi a^n} r^k \sin k\theta$$

# Question 14

$$a^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$D = \frac{1}{a^2} \qquad D = [\text{length}]^2 / \text{time}$$

$$\begin{cases} u(0,t) = A \\ u(L,t) = B \\ u(x,0) = x^2(L-x) \end{cases} \longrightarrow \quad y = \frac{x}{L} \qquad s = \frac{t}{\tau} \qquad \tau = \frac{L^2}{D} = L^2 a^2$$

$$\left( z(y,s) = \frac{u(yL, s\tau) - A}{B - A} \right.$$

$$\begin{cases} z(0,s) = 0 \\ z(1,s) = 1 \end{cases}$$

$$u(x,t) = z\left(\frac{x}{L}, \frac{t}{\tau}\right)(B-A) + A$$

## 1) Dimensionless equation

$$a^2 \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial t} = \frac{\partial z}{\partial s} \cdot \frac{1}{\tau}(B-A)$$

$$\frac{\partial u}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{1}{L}(B-A)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}\left(\frac{1}{L}\right)^2 (B-A)$$

$$a^2 \frac{\partial z}{\partial s} \frac{1}{\tau}(B-A) - \frac{\partial^2 z}{\partial y^2}\left(\frac{1}{L}\right)^2 (B-A) \qquad \tau = L^2 a^2$$

$$\rightarrow \frac{\partial z}{\partial s}(B-A) - \frac{\partial^2 z}{\partial y^2}(B-A) = 0$$

Part I (steady state)

$$\cancel{\frac{\partial z}{\partial s}} - \frac{\partial^2 z}{\partial y^2} = 0 \rightarrow$$

steady state solution

$$\frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow z^{st} = Ay + B$$

using BC's $\begin{cases} B = 0 \\ A = 1 \end{cases}$

$\Rightarrow$ steady state solution : $z^{st}(y)$

Part II → transient solution

$$U(x,s) = z^{st} - z(x,s) = \underbrace{y - z(x,s)}$$

Boundary conditions on $U(x,t)$

$z(0,s) = 0$ $\quad$ $U(0,s) = 0$

$z(1,s) = 1$ $\quad$ $U(1,s) = 0$

↳ take full time varying linear equation

$$\frac{\partial U}{\partial s} - \frac{\partial^2 U}{\partial y^2} = 0$$

→ $U(y,s) = v(y)\,w(s)$

→ $w'(s)\,v(y) - v''(y)\,w(s) = 0$

$$\frac{W'(s)}{W(s)} = \frac{v''(y)}{v(y)} = \lambda \longrightarrow$$

$\lambda = 0 \rightarrow$

$\lambda > 0 \rightarrow$

$\lambda < 0 \rightarrow$

$$U(y,s) = \sum_{k=1}^{\infty} A_k e^{-ks} \sin ky$$

$$u(x,0) = x^2(L-x) \rightarrow z(y,0) = \frac{\overbrace{u(yL,0)} - A}{B-A} + A$$

$$z(y,0) = \frac{\underbrace{(yL)^2(L-yL)} - A}{B-A} + A$$

$$U(y,0) = y - z(y,0) = y - \underbrace{\left[\frac{(yL)^2(L-yL)}{B-A} + A\right]}$$

$$U(y,0) = \sum_{k=1}^{\infty} A_k \sin ky = \underbrace{y - \left[\frac{(yL)^2(L-yL)}{B-A} + A\right]}_{h(y)}$$

next steps: build the odd function on $[-L, L]$

that is $h(y)$ on $[0, L]$ or $-h(y)$ on $[-L, 0]$

then compute Fourier series

then match with expression of $U(y, 0)$

Final solution $z(y, s) = U(y, s) + z^{st}$

$= U(y, s) + y$