MATH-UA 9263 - Partial Differential Equations Recitation 6: Transport equations and Method of characteristics

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Question 1 *Solve the first order equation* $2u_t + 3u_x = 0$ *with the auxiliary condition* $u = \sin x$ *when* $t = 0$

Question 2 *We consider the following PDE for* $u(x, y)$*:*

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = R$$

We consider the Cauchy problem with the initial curve defined by $x(s) = s$ *and* $y(s) = s$ *(with* $s > 0$*). On this curve, we specify the following function*

$$u(s, s) = f(s)$$

for a given $f(s)$*.*

a) Derive the set of characteristics and sketch the curves in the region $x > 0, y > 0$*. Verify that the Cauchy problem is well posed.*

b) Compute the solution for the following cases

- *i)* $u(x, y)$ *for* $R = 0$
- *ii)* $R = U$ *(*U *constant)*
- *iii)* $R = U\frac{x}{L}$ *(*U *and* L *constant)*
- *iv)* $R = U\frac{xy}{L^2}$ *(*U *and* L *constants)*

Question 3 *Solve* $au_x + bu_y = 0$ *where* a, b *are constants,* $a \neq 0$*, with the initial condition* $u(0, y) = e^y$

Question 4 *We consider the transport equation for* $u(x, t)$

$$\frac{\partial}{\partial x}(cu) + \frac{\partial u}{\partial t} = 0$$

with

$$c = c(x) = c_0 \frac{x^2}{(x^2 + L^2)}$$

where c_0 *and* L *are constants with respectively speed and length units. We consider the initial value problem for which the initial curve is defined as* $x(s) = s$ *with* $s \geq 0$ *and* $t(s) = 0$ *on which we specify the function*

$$u(s, 0) = f(s) = Ue^{-s/L}$$

where U *is constant.*

a) *Sketch* $c(x)$. *Assuming* $\lim_{x\to\pm\infty} u(x,t) = 0$, *using the PDE and the expression of* $c(x)$, *show that the integral* $\int_0^\infty u(x,t)\ dt$ *is conserved through time.*

b) *Obtain and sketch (for* $x \geq 0$, $t \geq 0$*) the set of characteristics. Check that the Cauchy problem is well posed.*

c) *Explain why we can't enforce a boundary condition of the form* $u(0,t) = h(t)$ *at* $x = 0$

d) *Compute the solution* $u(x,t)$

e) *Sketch the solution* u/U *as a function of* x/L *and for a couple of distinct time steps* $c_0 t/L$. *Now that you have the expression of* $u(x,t)$, *make sure that the integral from a) is conserved.*

Question 5 *Solve* $u_x + yu_y = y^2$ *with the initial condition* $u(0, y) = \sin y$

Question 6 *Solve* $u_x + yu_y = u^2$ *with the initial condition* $u(0, y) = \sin y$

Question 7 (Burger's equation) *Solve* $u_t + uu_x = -x$, $u(x, 0) = \phi(x)$.

Question 8 *Solve the quasilinear initial value problem*

$$(y + u)u_x + yu_y = x - y, \quad u(x, 1) = 1 + x$$

Question 9 a) *Solve the equation*

$$u_t + \left(\frac{u^3}{3}\right)_x = 0$$

for $t > 0$, $-\infty < x < \infty$ *with initial data*

$$u(x, 0) = h(x) = \begin{cases} -a(a - e^x) & x < 0 \\ -a(a - e^{-x}) & x > 0 \end{cases}$$

where $a > 0$ *is constant. Solve until the first appearance of discontinuous derivative and determine the critical time.*

b) Consider the equation

$$u_t + \left(\frac{u^3}{3}\right)_x = -cu$$

How large does the constant $c > 0$ have to be so that a smooth solution (with no discontinuities) exists for all $t > 0$? Explain.

Question 10 *Solve the equation $(1 + x^2)u_x + u_y = 0$. Sketch some of the characteristic curves*

Question 11 *Using Duhamel's method, solve the problem*

$$\begin{cases} c_t + vc_x = f(x,t) & x \in \mathbb{R}, t > 0 \\ c(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

Find an explicit formula when $f(x,t) = e^{-t}\sin x$ [Hint: for a fixed $s \geq 0$ and $t > s$, solve

$$\begin{cases} w_t + vw_x = 0 \\ w(x,s;s) = f(x,s) \end{cases}$$

and integrate w with respect to s over $(0,t)$.]

Question 12 *Determine the solution of $\frac{\partial \rho}{\partial t} = \rho$ that satisfies $\rho(x,t) = 1 + \sin x$ along $x = -2t$.*

Question 13 *Consider the traffic flow problem*

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$$

Assume $u(\rho) = u_{max}(1 - \rho/\rho_{max})$. Solve for $\rho(x,t)$ if the initial conditions are

a) $\rho(x,0) = \rho_{max}$ for $x < 0$ and $\rho(x,0) = 0$ for $x > 0$. This corresponds

Question 14 *Consider the following problem ($a > 0$):*

$$\begin{cases} u_t + au_x = f(x,t) & 0 < x < R, t > 0 \\ u(0,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < R \end{cases}$$

Prove the stability estimate

$$\int_0^R u^2(x,t)\,dx \leq e^t \int_0^t \int_0^R f^2(x,s)\,dx\,ds, \quad t > 0$$

[Hint: Multiply by u the equation. Use $a > 0$ and the inequality $2fu \leq f^2 + u^2$ to obtain

$$\frac{d}{dt}\int_0^R u^2(x,t)\,dx \leq \int_0^R f^2(x,t)\,dx + \int_0^R u^2(x,t)\,dx$$

Prove that if $E(t)$ satisfies $E'(t) \leq G(t) + E(t)$, $E(0) = 0$ then $E(t) \leq e^t \int_0^t G(s)\,ds$]

References

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[3] Walter A. Strauss, *Partial Differential Equations An Introduction*, John Wiley and Sons Ltd, 2008

[4] Sandro Salsa, *Partial Differential Equations in Action, From Modelling to Theory*, Springer, 2016.

[5] Lawrence C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, 2010.