MATH-UA 9263 - Partial Differential Equations Recitation 8: Non Linear first order equations + Wave equation (part I)

Augustin Cosse

January 2022

Question 1 *The Helmholtz equation can be obtained as the Fourier transform of the wave equation,*

$$\Delta u + \frac{\omega^2}{c^2} u = 0$$

1. *At high frequency, this equation can be approximated by the* eikonal equation. *To see this,*
   (a) *first write the solution as the wave*

   $$u(\boldsymbol{x}, \omega) = A(\boldsymbol{x}, \omega) e^{i\omega\phi(\boldsymbol{x})}$$

   (b) *Using this, show that the Helmholz equation reduces to*

   $$\left((\partial_j \phi)^2 - \frac{1}{c^2}\right) - \frac{i}{\omega}\left(\frac{2}{A}\partial_j A \partial_j \phi + \partial_j^2 \phi\right) - \frac{1}{\omega^2 A}\partial_j^2 A = 0$$

   (c) *Finally take the limit $\omega \to \infty$ and show that the previous equation reduces to*

   $$|\nabla \phi(\boldsymbol{x})|^2 = \frac{1}{c^2} \tag{1}$$

2. *Solve the eikonal equation with the initial condition $\phi(x, 0) = -\sqrt{1 + x^2}$*
3. *The wavefront of a time-varying field can be defined as the set of all the points where the wave has the same phase. Knowing that light propagates in the direction of the gradient of the phase, show that light always propagates normal to the wavefront.*

Question 2 *We want to solve the equation*

$$u = u_x^2 - 3u_y^2$$

with the initial condition $u(x, 0) = x^2$

Question 3 *Solve the Cauchy problem*

$$\begin{cases} u_x = -(u_y)^2 & x > 0, y \in \mathbb{R} \\ u(0,y) = 3y & y \in \mathbb{R} \end{cases}$$

Question 4 *We consider an elementary incoming plane wave* $e^{i(\boldsymbol{k}_I \cdot \boldsymbol{x} - \omega t)}$ *where* $\boldsymbol{k}$ *represents the constant incoming wave number. We assume that this plane wave reflects off a curved boundary which is parametrized as* $(x,y) = (x_0(\tau), y_0(\tau))$*. Let* $R(x,y)e^{i(u(x,y)-\omega t)}$ *denote the reflected wave. We assume that the total field on the interface is zero (this implies* $0 = e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} + R(x,y)e^{i\phi(\boldsymbol{x})}e^{-i\omega t}$*) and hence* $u(x_0,y_0) = \boldsymbol{k} \cdot \boldsymbol{x}_0$*. Using this, together with the fact that light propagates in the direction* $\nabla\phi(\boldsymbol{x})$ *and that the phase of the reflected wave has to obey the eikonal equation*

$$|\phi_x|^2 + |\phi_y|^2 = |\boldsymbol{k}_I|$$

show that the angle of reflection off a curved boundary is the same as the angle of incidence.

Question 5 *Solve the following Cauchy problem*

$$\begin{cases} u_x^2 + u_y^2 = 4u & (x,y) \in \mathbb{R}^2 \\ u(x,-1) = x^2 & x \in \mathbb{R} \end{cases}$$

Question 6 *Solve the Cauchy problem*

$$\begin{cases} c^2(u_x^2 + u_y^2) = 1, & (x,y) \in \mathbb{R}^2 \\ u(\cos s, \sin s) = 0, & s \in \mathbb{R} \end{cases}$$

Question 7 *We consider the transverse vibrations of a simply supported beam which has zero initial velocity and whose initial displacement is* $f(x) = Ax(L-x)$ *where* A *is a constant and* L *is the length of the beam. This phenomenon can be described by the fourth order PDE*

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0$$

say with the boundary conditions

1. $y(0,t) = 0$
2. $y(L,t) = 0$
3. $\frac{\partial^2}{\partial x^2} y(0,t) = 0$
4. $\frac{\partial^2}{\partial x^2} y(L,t) = 0$
5. $\frac{\partial}{\partial t} y(x,0) = 0$
6. $y(x,0) = Ax(L-x)$

Find the solution of this problem.

Question 8 *Solve the vibrating string problem for the following boundary conditions*

1. $u(0,t) = 0$
2. $u(1,t) = 0$
3. $\frac{\partial}{\partial t}u(x,0) = \frac{x(x-1)}{100}$
4. $u(x,0) = \begin{cases} \frac{x}{100} & 0 < x < 1/2 \\ \frac{1-x}{100} & 1/2 < x < 1 \end{cases}$

Question 9 *Solve the vibrating string problem if the string (of length L) if fixed at the end points, and has initial velocity given by*

$$u(x,0) = \begin{cases} \frac{h}{\alpha L}x & x < \alpha L \\ -\frac{h}{L(1-\alpha)}x + \frac{h}{L(1-\alpha)}\alpha L + h & x > \alpha L \end{cases}$$

and initial displacement given by

$$\frac{\partial}{\partial t}u(x,0) = \begin{cases} \frac{4k}{L}x & x < 1/4L \\ k & 1/4L < x \leq 3/4L \\ -\frac{4k}{L}x + \frac{3k}{+}k & \end{cases}$$

For $\alpha = 1/4$

References

[1] Kenneth S. Miller, *Partial Differential Equations in Engineering Problems*, Dover Publications inc. 2020.

[2] Walter A. Strauss, *Partial Differential Equations An Introduction*, John Wiley and Sons Ltd, 2008

[3] Sandro Salsa, *Partial Differential Equations in Action, From Modelling to Theory*, Springer, 2016.

[4] Lawrence C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, 2010.