Partial Differential Equations, lecture 6

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Laplace and Poisson's equations

Laplace's equation $\Delta u = 0$ occurs frequently in the applied sciences, in particular in the study of steady state phenomena. its solutions are called harmonic functions.

To be precise we say that a function u is harmonic in a domain $\Omega \subseteq \mathbb{R}^n$ if $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω.

As an example, the equilibrium position of an elastic membrane is a harmonic function as is the velocity of a homogeneous fluid. Another example is the steady state temperature of a homogeneous and isotropic body as we saw when studying the heat equation.

Slightly more general, Poisson's equation plays a role in the theory of conservative fields where the vector field is derived from the gradient of a potential. Examples include

- Electrostatics. If we consider a potential V, the relation between the electric field E and the potential is given by $E = -\text{grad } V$ (the electric field is perpendicular to the equipotentials). If D denotes the electric flux density, $D = \varepsilon E$, from the point form of Gauss law, we can write $\nabla \cdot D = \rho_v$ where ρ_v is the volume charge density. Then using the fact that E is derived from the potential V, we get

$$\nabla \cdot (-\varepsilon \nabla V) = \rho_v$$

  which for a constant permittivity ε can read as $-\Delta V = \rho_v/\varepsilon$

- Similarly, if we consider a gravitational field $F(\boldsymbol{x})$ generated from a mass density $\rho(\boldsymbol{x})$, and if this field can be expressed from the gravitational potential $\Phi(\boldsymbol{x})$, Gauss law for gravity is used to express the fact that the flux of the gravitational field F through the surface $S = \partial V$ is equal to $-4\pi G m$ where $m = \int_V \rho(\boldsymbol{x})\, d\boldsymbol{x}$ is the total mass contained in the volume V, i.e.

$$\oint_S F \cdot \vec{n}\, dS = -4\pi G m$$

From the assumption that the field F can be derived from a potential Φ, as in the electrostatic case, we then get

$$-\oint_S \nabla\Phi \cdot \vec{n}\, dS = -4\pi G m = -4\pi G \int_V \rho(\boldsymbol{x})\, d\boldsymbol{x}$$

and applying the divergence theorem,

$$\int_V \nabla \cdot (\nabla\Phi)\ dV = 4\pi G \int_V \rho(\boldsymbol{x})\, dV$$

As the relation holds for all volumes V, we can infer

$$\Delta\Phi = \nabla \cdot (\nabla\Phi) = 4\pi G\rho$$

In order to understand the properties of harmonic functions, let us consider a multidimensional random walk. That is to say, we fix a time step $T > 0$, a space step $h > 0$ and use $h\mathbb{Z}^2$ to denote the lattice of points $\boldsymbol{x} = (x_1, x_2)$ whose coordinates are integer multiples of h. If we use $p = p(x_1, x_2, t)$ to denote the transition probability function, the probability to find the the particle at position $\boldsymbol{x}$ at time $t + \tau$ can read as

$$p(\boldsymbol{x}, t+\tau) = \frac{1}{4}\left\{p(\boldsymbol{x} + h\boldsymbol{e}_1, t) + p(\boldsymbol{x} - h\boldsymbol{e}_1, t) + p(\boldsymbol{x} + h\boldsymbol{e}_2, t) + p(\boldsymbol{x} - h\boldsymbol{e}_2, t)\right\} \quad (1)$$

If we introduce the mean value operator M_h whose action on a function u is defined as

$$\begin{aligned} M_h u(\boldsymbol{x}) &= \frac{1}{4}\left\{u(\boldsymbol{x} + h\boldsymbol{e}_1) + u(\boldsymbol{x} - h\boldsymbol{e}_1) + u(\boldsymbol{x} + h\boldsymbol{e}_2) + u(\boldsymbol{x} - h\boldsymbol{e}_2)\right\} \\ &= \frac{1}{4} \sum_{|\boldsymbol{x}-\boldsymbol{y}|=h} u(\boldsymbol{y}) \end{aligned}$$

we get $p(\boldsymbol{x}, t+\tau) = M_h p(\boldsymbol{x}, t)$. Note that $M_h u(\boldsymbol{x})$ gives the average of u over the points of the lattice $h\mathbb{Z}^2$ at distance h from $\boldsymbol{x}$ (discrete neighborhood of $\boldsymbol{x}$ of radius h). Also note that the value of p at time $t+\tau$ is determined by the action of M_h on p at the previous time. For this reason, we call M_h the generator of the random walk. If u is twice continuously differentiable, taking the limit, we get

$$\lim_{h\to 0} \frac{M_h u(\boldsymbol{x}) - u(\boldsymbol{x})}{h^2} \to \frac{1}{4}\Delta u(\boldsymbol{x})$$

I.e. using a Taylor expansion around $\boldsymbol{x}$, for each of the $u(\boldsymbol{x} \pm h\boldsymbol{e}_j)$, we get

$$M_h u = u(\boldsymbol{x}) + \frac{h^2}{4} u_{x_1 x_1} + \frac{h^2}{4} u_{x_2 x_2} + O(h^3).$$

Such a formula suggests to introduce for any fixed $h > 0$ a discrete Laplace operator of the form

$$\Delta_h^* = \frac{4}{h^2}(M_h - I)$$

The operator Δ_h^* acts on functions u defined on the lattice $h\mathbb{Z}^2$ and we therefore say that a function u is d-harmonic (the d standing for *discrete*) if $\Delta_h^* u = 0$

We see that the value of a d-harmonic function at any point $\boldsymbol{x}$ is given by the average of the values of u at the points in the discrete neighborhood of $\boldsymbol{x}$ of radius h.

Guided by this discrete characterization, we will now establish fundamental properties of harmonic functions. Since d-harmonic functions are defined through a mean value property, it seems natural to expect that harmonic functions will inherit a similar mean value property. We can in fact say more as indicated by the following theorem

Theorem 1. *Let u be harmonic in $\Omega \subseteq \mathbb{R}^n$. Then for any ball $B_R(x) \subset\subset \Omega$*[a]*, the following mean value formulas hold*

$$u(\boldsymbol{x}) = \frac{n}{\omega_n R^n}\int_{B_R(\boldsymbol{x})} u(\boldsymbol{y})\, d\boldsymbol{y} \tag{2}$$

$$u(\boldsymbol{x}) = \frac{1}{\omega_n R^{n-1}}\int_{\partial B_R(\boldsymbol{x})} u(\boldsymbol{\sigma})\, d\boldsymbol{\sigma} \tag{3}$$

Where ω_n is the surface measure of ∂B_1 and ∂B_1 is the unit sphere in $\mathbb{R}^n$. In general $\omega_n = \frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ where $\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}\, dt$ is the Euler Gamma function.

[a] For any two open subsets U and V of $\mathbb{R}^n$, we write $U \subset\subset V$ and say that U is compactly contained in V if $U \subset \overline{U} \subset V$ and $\overline{U}$ is compact

Proof. We start with the second formula. For $r < R$ (note that $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{\sigma} \in \mathbb{R}^n$), we let

$$g(r) = \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{\sigma})\, d\boldsymbol{\sigma}$$

We apply the change of variables $\boldsymbol{\sigma} = \boldsymbol{x} + r\boldsymbol{\sigma}'$ (i.e $\boldsymbol{\sigma}' = \frac{\boldsymbol{\sigma}-\boldsymbol{x}}{r}$), then $\boldsymbol{\sigma}' \in \partial B_1(0)$, $d\boldsymbol{\sigma} = r^{n-1}d\boldsymbol{\sigma}'$ and

$$g(r) = \frac{1}{\omega_n}\int_{\partial B_1(0)} u(\boldsymbol{x} + r\boldsymbol{\sigma}')\, d\boldsymbol{\sigma}'$$

Let $v(\boldsymbol{y}) = u(\boldsymbol{x} + r\boldsymbol{y})$. From this we have

$$\nabla_y v(\boldsymbol{y}) = r\nabla u(\boldsymbol{x} + r\boldsymbol{y})$$
$$\Delta_y v(\boldsymbol{y}) = r^2 \Delta u(\boldsymbol{x} + r\boldsymbol{y})$$

Then

$$g'(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} \frac{d}{dr} u(\boldsymbol{x} + r\boldsymbol{\sigma}') d\boldsymbol{\sigma}' = \frac{1}{\omega_n} \int_{\partial B_1(0)} \nabla_\xi u(\boldsymbol{x} + r\boldsymbol{\sigma}') \cdot \boldsymbol{\sigma}' \, d\boldsymbol{\sigma}'$$
$$= \frac{1}{\omega_n} \frac{1}{r} \int_{\partial B_1(0)} \nabla_{\sigma'} v(\boldsymbol{\sigma}') \cdot \boldsymbol{\sigma}' \, d\boldsymbol{\sigma}'$$

Using the divergence theorem, we obtain

$$\frac{1}{\omega_n r} \int_{\partial B_1(0)} \nabla v(\sigma') \cdot \boldsymbol{\sigma}' \, d\boldsymbol{\sigma}' = \frac{1}{\omega_n r} \int_{B_1(0)} \Delta v(\boldsymbol{y}) \, d\boldsymbol{y} = \frac{r}{\omega_n} \int_{B_1(0)} \Delta u(\boldsymbol{x} + r\boldsymbol{y}) \, d\boldsymbol{y} = 0$$

In the last equality we use the fact that the function u is harmonic. This shows that the function $g(r)$ is constant (i.e. $g'(r) = 0$) yet since $g(r) \to u(\boldsymbol{x})$ as $r \to 0$, we must have $g(r) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(\boldsymbol{x})} u(\boldsymbol{\sigma}) \, d\boldsymbol{\sigma} = u(\boldsymbol{x})$ for every R.

To derive the second relation, simply note that using

$$u(\boldsymbol{x}) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{\sigma}) \, d\boldsymbol{\sigma},$$

multiplying by r^{n-1} and integrating both sides over $r = 0$ to R we get

$$\int_0^R r^{n-1} u(\boldsymbol{x}) \, d\boldsymbol{x} = \frac{1}{\omega_n} \int_0^R dr \int_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{\sigma}) \, d\boldsymbol{\sigma}$$
$$\Leftrightarrow \frac{R^n}{n} u(\boldsymbol{x}) = \frac{1}{\omega_n} \int_{B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, d\boldsymbol{y}$$

This concludes the proof. □

We say that a continuous function satisfies the mean value property in Ω if the two relations (2) and (3) hold for any ball $B_R(\boldsymbol{x}) \subset\subset \Omega$. In fact, one can show that if u is continuous and possesses the mean value property in a domain Ω, then u is harmonic in Ω. From this, we see that there exists an equivalent characterization of harmonic functions through the mean value property.

Using the mean value formula, we can derive the following maximum principle known as Harnack's inequality

Theorem 2 (Harnack's inequality)**.** *For each connected open set $V \subset\subset U$, there exists a positive constant c depending only on V such that*

$$\sup_V u \leq c \inf_V u$$

For all non negative harmonic functions u in U.

What Harnack's inequality is demonstrating is that the values of non negative harmonic functions are all comparable. I.e. $u(\boldsymbol{x})$ cannot be very small (or very large) at any point in V unless it is very small (or very large) in all V.

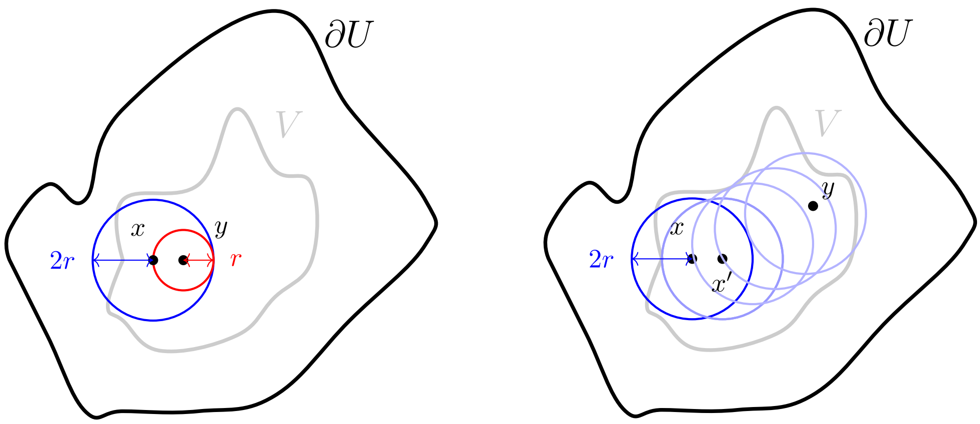

Figure 1: For any two points $x, y \in V$, the set V can be covered by a sequence of balls of sufficiently small radius r ensuring that the non negative function $u(\boldsymbol{x})$ remains harmonic inside the ball. One can then relate the points inside each ball by means of the mean value formulas.

This follows from the fact that for any $\boldsymbol{x}$ that does not lie on the boundary, one can always find a small neighborhood in which u satisfies Laplace's equation and thus the mean value formulas.

Proof. Let $V_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ denote the volume of the unit ball in $\mathbb{R}^n$. Note that this implies $|B_r| = V_n r^n$ (i.e. the volume of the radius r ball is obtained by multiplying V_n by r^n)

Take $r = \frac{1}{4}\text{dist}\,(V, \partial U)$ and choose $x, y \in V$ with $|x - y| \leq r$. From the mean value formula, we get

$$u(\boldsymbol{x}) = ⨍_{B(\boldsymbol{x},2r)} u(z)\, dz \geq \frac{1}{V_n 2^n r^n} \int_{B(\boldsymbol{y},r)} u(z)\, dz = \frac{1}{2^n} ⨍_{B(\boldsymbol{y},r)} u(z)\, dz = \frac{1}{2^n} u(\boldsymbol{y})$$

As this relation holds for any pair $(\boldsymbol{x}, \boldsymbol{y})$, we can replace $\boldsymbol{x}$ with $\boldsymbol{y}$ (and vice versa) from which we get

$$2^n u(\boldsymbol{y}) \geq u(\boldsymbol{x}) \geq \frac{1}{2^n} u(\boldsymbol{y})$$

Since V is connected and $\overline{V}$ is compact, we can cover $\overline{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius $2r$ and such that $B_i \cap B_{i-1} \neq \emptyset$ for $i = 2, \ldots N$ (Fig. 1). Then

$$u(\boldsymbol{x}) \geq \frac{1}{2^{n(N+1)}} u(\boldsymbol{y}), \quad \forall \boldsymbol{x}, \boldsymbol{y} \in V$$

□

Fundamental solution and invariant properties

As we did for the diffusion equation, let us start by looking at the invariant properties characterizing the operator Δ. Those include

- Translations. Clearly, if $u(\boldsymbol{x})$ is harmonic, we have $u(\boldsymbol{x}-\boldsymbol{y})$ harmonic for any fixed $\boldsymbol{y}$.
- Rotations. Invariance by rotation means that given a rotation in $\mathbb{R}^n$, represented by an orthogonal matrix $\boldsymbol{M}$, (i.e. such that $\boldsymbol{M}^{-1}=\boldsymbol{M}^T$), $v(\boldsymbol{x})=u(\boldsymbol{M}\boldsymbol{x})$ is also harmonic in $\mathbb{R}^n$. To check this, observe that if we denote by D^2u the Hessian of u (encoding the second order derivatives), we have $\Delta u = \mathrm{Tr}(D^2u)$. In particular, since $D^2v(\boldsymbol{x}) = \boldsymbol{M}^T D^2 u(\boldsymbol{M}\boldsymbol{x})\boldsymbol{M}$, one can show $\mathrm{Tr}(\boldsymbol{M}^T D^2 u(\boldsymbol{M}\boldsymbol{x})\boldsymbol{M}) = \mathrm{Tr}\left(D^2u(\boldsymbol{M}\boldsymbol{x})\right) = \Delta u = 0$ (as $\boldsymbol{M}$ is orthogonal). Now, as for the heat equation, one could wonder if the invariance does not arise from the particular structure of the solutions. In this case, a typical rotation invariant quantity is the distance function from a point, that is $r=|\boldsymbol{x}|$. In this case, it therefore makes sense to look for a solution of the form $u=u(r)$. Such a solution is derived below

Substituting $u=u(r)$ in Laplace's equation and using the cylindrical and spherical formulations of the Laplacian, we get

- In dimension 2, $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} = 0$ so that $u(r) = C_1 \log r + C_2$.
- In dimension 3, using spherical coordinates r, ψ, θ, $r>0$, $0 \le \psi < \pi$, $0 \le \theta < 2\pi$, the Laplacian reads as

  $$\Delta = \underbrace{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{f\partial}{\partial r}}_{\text{Radial part}} + \underbrace{\frac{1}{r^2}\left\{\frac{1}{(\sin\psi)^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\psi^2} + \cot\psi\frac{\partial}{\partial\psi}\right\}}_{\substack{\text{Spherical Part} \\ \text{Laplace Beltrami operator}}}$$

  For a function that only depends on r, we see that again, Laplace's equation reduces to

  $$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = 0$$

  The general solution of this equation is $u(r) = \frac{C_1}{r} + C_2$, where C_1, C_2 are arbitrary constants. Choosing $C_2 = 0$ and $C_1 = (4\pi)^{-1}$ if $n=3$, $C_1 = -(2\pi)^{-1}$ if $n=2$, the function

$$\Phi(\boldsymbol{x}) = \begin{cases} -\dfrac{1}{2\pi}\log|\boldsymbol{x}| & n=2 \\ \dfrac{1}{n(n-1)V(n)|\boldsymbol{x}|^{n-2}} & n\geq 3 \end{cases}$$

is called the fundamental solution of Laplace's equation. Note that $V(n)$ encodes the volume of the radius one ball in $\mathbb{R}^n$. The particular choice of constants is made to satisfy $\Delta\Phi(\boldsymbol{x}) = -\delta_n(\boldsymbol{x})$ where $\delta_n(\boldsymbol{x})$ is the n-dimensional Dirac measure that we introduced in previous lectures. When $n = 3$, $4\pi\Phi$ represents the electrostatic potential due to a unitary charge located at the origin and vanishing at ∞.

The Newtonian potential

Suppose that $(4\pi)^{-1}f(\boldsymbol{x})$ gives the density of charge inside a compact set in $\mathbb{R}^3$. Then $\Phi(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}$ represents the potential at $\boldsymbol{x}$ due to the charge $(4\pi)^{-1}f(\boldsymbol{y})\,d\boldsymbol{y}$ inside a small region of volume $d\boldsymbol{y}$ around $\boldsymbol{y}$. The full potential is given by the sum of all the contributions from the charges distributed according to $f(y)$

$$u(\boldsymbol{x}) = \int_{\mathbb{R}^3} \Phi(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y} = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\,d\boldsymbol{y} \tag{4}$$

which is the convolution between f and Φ and is called the Newtonian potential of f. Informally (we will show that this formally holds in the proof of Theorem 3 below) we have

$$\Delta u = \int_{\mathbb{R}^3} \Delta_{\boldsymbol{x}}\Phi(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y} = -\int_{\mathbb{R}^3} \delta_3(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y} = -f(\boldsymbol{x})$$

Note that (4) is not the only solution to $\Delta u = -f$ since $u(\boldsymbol{x}) + c$ is a solution for any constant c. However, the Newtonian potential is the only solution vanishing at ∞. This idea is summarized by the following theorem

Theorem 3. *Let $f \in C^2(\mathbb{R}^n)$ with compact support. Let u be the Newtonian potential of f defined by*

$$u(\boldsymbol{x}) = \int_{\mathbb{R}^n} \Phi(\boldsymbol{x}-\boldsymbol{y})\; f(\boldsymbol{y})\; d\boldsymbol{y} \tag{5}$$

Then u is the only solution in $\mathbb{R}^n$ of $\Delta u = -f$ belonging to $C^2(\mathbb{R}^n)$ and vanishing at ∞.

Proof. As usual when we want to prove a uniqueness result, we consider another solution $v \in C^2(\mathbb{R}^3)$ vanishing at infinity. Note that from an application of Theorem 2, we get

$$\sup(u-v) \leq c\inf(u-v)$$

Since both u and v vanish at ∞, this necessarily implies $u - v = 0$. In other words, as soon as we have a solution, we know that this solution is unique. we are thus left with showing that the solution u corresponding to the Newtonian potential satisfies Poisson's equation and is $C^2(\mathbb{R}^3)$. Recall that we have

$$\Phi(\boldsymbol{x}) = \begin{cases} -\dfrac{1}{2\pi}\log|\boldsymbol{x}| & n=2 \\ \dfrac{1}{n(n-2)V(n)}\dfrac{1}{|\boldsymbol{x}|^{n-2}} & n\geq 3 \end{cases}$$

Now note that

$$u(\boldsymbol{x}) = \int_{\mathbb{R}^3} \Phi(\boldsymbol{y})f(\boldsymbol{y}-\boldsymbol{x})\, d\boldsymbol{y} = \int_{\mathbb{R}^3} f(\boldsymbol{x}-\boldsymbol{y})\Phi(\boldsymbol{y})\, d\boldsymbol{y}$$

To show that u is C^2 we consider the ratios

$$\frac{u(\boldsymbol{x}+h\boldsymbol{e}_i)-u(\boldsymbol{x})}{h} = \int_{\mathbb{R}^3} \Phi(\boldsymbol{y})\left[\frac{f(\boldsymbol{x}+h\boldsymbol{e}_i-\boldsymbol{y})-f(\boldsymbol{x}-\boldsymbol{y})}{h}\right]\, d\boldsymbol{y}$$

Now in order to take the limit and move it inside the integral, we need to satisfy two conditions: (i) uniform convergence of the sequence

$$g_h \equiv \left[\frac{f(\boldsymbol{x}+h\boldsymbol{e}_i-\boldsymbol{y})-f(\boldsymbol{x}-\boldsymbol{y})}{h}\right]$$

and (ii) integration over a finite set. Condition (ii) is satisfied since $f(\boldsymbol{x})$ has finite support. For the uniform convergence, note that for the first order derivatives, we can always use a Taylor approximation, which gives

$$\left|\frac{\partial f}{\partial x_i} - \frac{f(\boldsymbol{x}+h\boldsymbol{e}_i-\boldsymbol{y})-f(\boldsymbol{x}-\boldsymbol{y})}{h}\right| \leq h\sup|D^2 f|$$

The right-hand side (which does not depend on $\boldsymbol{x}$) is well defined since by assumption f is twice continuously differentiable. For the uniform continuity of the sequence of approximations of the second order partial derivatives, this is a little trickier. We can still derive a similar result by relying on the fact that f has compact support. As any continuous function on a compact set is automatically uniformly continuous, we can apply the mean value theorem (the first order derivatives are all continuous and differentiable) and we thus have

$$\frac{1}{h}\left[\frac{\partial f}{\partial x_i}(\boldsymbol{x}+h\boldsymbol{e}_j-\boldsymbol{y}) - \frac{\partial f}{\partial x_i}(\boldsymbol{x}-\boldsymbol{y})\right] = \frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}+th\boldsymbol{e}_j-\boldsymbol{y})$$

for some $t\in[0,1]$. From this,

$$\begin{aligned}&\left|\frac{1}{h}\left\{\frac{\partial f}{\partial x_i}(\boldsymbol{x}+h\boldsymbol{e}_j-\boldsymbol{y}) - \frac{\partial f}{\partial x_i}(\boldsymbol{x}-\boldsymbol{y})\right\} - \frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}-\boldsymbol{y})\right| \\ &= \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}+th\boldsymbol{e}_j-\boldsymbol{y}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}-\boldsymbol{y})\right|\end{aligned}$$

In particular, since the mean value theorem holds for any value of $\boldsymbol{x}$ and since the derivatives $\frac{\partial^2}{\partial x_i \partial x_j}$ are uniformly continuous, we have that for all ε there exist H such that for all $h<H$, for all $\boldsymbol{x}$,

$$\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}+th\boldsymbol{e}_j-\boldsymbol{y}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}-\boldsymbol{y})\right| < \varepsilon$$

and hence

$$\left|\frac{1}{h}\left\{\frac{\partial f}{\partial x_i}(\boldsymbol{x}+th\boldsymbol{e}_j-\boldsymbol{y})-\frac{\partial f}{\partial x_i}(\boldsymbol{x}-\boldsymbol{y})\right\}-\frac{\partial^2 f}{\partial x_i\partial x_j}(\boldsymbol{x}-\boldsymbol{y})\right|<\varepsilon$$

which shows that the sequence of approximations

$$\frac{1}{h}\left\{\frac{\partial f}{\partial x_i}(\boldsymbol{x}+h\boldsymbol{e}_j-\boldsymbol{y})-\frac{\partial f}{\partial x_i}(\boldsymbol{x}-\boldsymbol{y})\right\}$$

converges uniformly to $\frac{\partial^2 f}{\partial x_i\partial x_j}$. From this, we can thus move the derivatives inside the integral which gives

$$\frac{\partial^2 u}{\partial x_i\partial x_j}=\int_{\mathbb{R}^n}\Phi(\boldsymbol{y})\frac{\partial^2 f}{\partial x_i\partial x_j}(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{y}$$

The derivatives $\frac{\partial^2 u}{\partial x_i\partial x_j}$ are continuous because the function f is C^2 by assumption. Note that the fundamental solution $\Phi(\boldsymbol{y})$ is defined everywhere except at 0. We can however still compute the integral (which should be understood as an improper integral) as

$$\int_{\mathbb{R}^n}\Phi(\boldsymbol{y})\partial_{x_ix_j}(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{y}=\lim_{\varepsilon\to 0}\int_{R^n\setminus B_\varepsilon(0)}\Phi(\boldsymbol{y})\partial_{x_ix_j}(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{y}$$

If we consider the integral in that same improper sense, from the fact that f is C^2 on a compact set, we have that $\partial_{x_i,x_j}f$ are uniformly continuous and we can write that for all ε', there exists δ such that for all $|z|<\delta$,

$$\begin{aligned}
&\left|\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\Phi(\boldsymbol{y})\partial_{x_ix_j}f(\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{z})\,d\boldsymbol{y}-\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\Phi(\boldsymbol{y})\partial_{x_ix_j}f(\boldsymbol{x}-\boldsymbol{y})\,\boldsymbol{d}y\right|\\
&=\left|\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\Phi(\boldsymbol{y})\left[\partial_{x_ix_j}f(\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{z})-\partial_{x_ix_j}f(\boldsymbol{x}-\boldsymbol{y})\right]\,d\boldsymbol{y}\right|\\
&\le\sup_{y\in\mathbb{R}^n\setminus B_\varepsilon(0)}\left|\partial_{x_ix_j}f(\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{z})-\partial_{x_ix_j}f(\boldsymbol{x}-\boldsymbol{y})\right|\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\Phi(\boldsymbol{y})\,d\boldsymbol{y}\\
&\le\varepsilon'\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\Phi(\boldsymbol{y})\,d\boldsymbol{y},\quad\forall\,\boldsymbol{x}
\end{aligned}$$

□

which shows continuity of $\partial_{x_ix_j}u$.

We now need to show that u satisfies Poisson's equation. Again we interpret our integral in the improper sense as $\Phi(\boldsymbol{y})$ is not defined at 0. We let

$$\Delta u(\boldsymbol{x})=\int_{\mathbb{R}^n\setminus B(0,\varepsilon)}\Phi(\boldsymbol{y})\Delta_x f(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{y}$$

Integrating by parts gives

$$\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(\boldsymbol{y})\Delta_x f(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} = \int_{\partial B(0,\varepsilon)} \Phi(\boldsymbol{y}) \cdot \frac{\partial f}{\partial \nu}(\boldsymbol{x}-\boldsymbol{y})\, dS(\boldsymbol{y}) \tag{6}$$

$$- \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} D\Phi(\boldsymbol{y}) \cdot D_y f(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} \tag{7}$$

For (6) we have

$$(6) \leq \sup_{\partial B(0,\varepsilon)} |Df(\boldsymbol{x})| \int_{\partial B(0,\varepsilon)} |\Phi(\boldsymbol{y})|\, dS(\boldsymbol{y}) \leq \begin{cases} C\varepsilon \log|\varepsilon| & n=2 \\ C\varepsilon & n \geq 3 \end{cases}$$

For (7), further integrating by parts yields

$$\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} D\Phi(\boldsymbol{y}) \cdot D_y f(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} = -\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Delta\Phi(\boldsymbol{y}) f(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} + \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(\boldsymbol{y}) f(\boldsymbol{x}-\boldsymbol{y})\, dS(\boldsymbol{y})$$

Since Φ is harmonic on $\mathbb{R}^n \setminus 0$, this further simplifies into

$$(7) = -\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(\boldsymbol{y}) f(\boldsymbol{x}-\boldsymbol{y})\, dS(\boldsymbol{y})$$

To control this last term, note that, from the definition of the fundamental solution, we have

$$D\Phi(\boldsymbol{y}) = -\frac{1}{nV(n)} \frac{\boldsymbol{y}}{|\boldsymbol{y}|^n}$$

as well as

$$\boldsymbol{\nu} = \frac{\boldsymbol{y}}{|\boldsymbol{y}|} = \frac{-\boldsymbol{y}}{\varepsilon}$$

From those relations, we can write the derivative $\partial_\nu \Phi$ along the normal vector ν as

$$\frac{\partial \Phi}{\partial \nu} = \boldsymbol{\nu} \cdot D\Phi(\boldsymbol{y}) = \frac{|\boldsymbol{y}|^2}{nV(n)|\boldsymbol{y}|^{n+1}} = \frac{1}{nV(n)\varepsilon^{n-1}}$$

but the ball $B(0,\varepsilon)$ has a surface S precisely given by $nV(n)\varepsilon^{n-1}$. Hence we have

$$\begin{aligned}(7) &= -⨍_{\partial B(0,\varepsilon)} f(\boldsymbol{x}-\boldsymbol{y})\, dS(\boldsymbol{y}) = -⨍_{\partial B(\boldsymbol{x},\varepsilon)} f(\boldsymbol{y})\, dS(\boldsymbol{y}) \\ &= -f(\boldsymbol{x}) \quad \text{when } \varepsilon \to 0\end{aligned}$$

which concludes the proof.

The Green function

Divergence formula, integration by parts and Green formulas

For any set $U \subset \mathbb{R}^n$ with boundary ∂U, let $\boldsymbol{\nu}^i$ denote the i^{th} component (along x_i) of the outward pointing unit normal vector at a point x of the boundary ∂U. In particular, for any function $u \in C^1(\overline{U})$, we call

$$\frac{\partial u}{\partial \nu} = \boldsymbol{\nu} \cdot D\boldsymbol{u}$$

the (outward) normal derivative of $\boldsymbol{u}$.

The Green formulas which will be used in the derivation of the Green function can be proved from the Gauss-Green theorem which is given below

Theorem 4 (Gauss-Green)**.** *Let $u \in C^1(\overline{U})$. We have*

(i) $\int_U u_{x_i} dx = \int_{\partial U} \boldsymbol{u} \cdot \boldsymbol{\nu}^i \, dS$

(ii) Applying (i) to each component u_{x_i} and summing, we get

$$\int_U \text{div } \boldsymbol{u} = \int_{\partial U} \boldsymbol{u} \cdot \boldsymbol{\nu} \, dS \tag{8}$$

which is known as the Divergence Theorem which we used in earlier developments.

Applying part (i) of the Gauss-Green Theorem to the product (u, v) of any two functions u and v gives the integration by parts formula:

$$\int_U u_{x_i} v \, dx = -\int_U u v_{x_i} \, dx + \int_{\partial U} (uv) \, \nu^i \, dS \tag{9}$$

Finally, the Green formulas are obtained by using integration by parts on u_{x_i} instead of u, taking $v = 1$,

$$\int_U \Delta u \, dx = \int_{\partial U} \sum_{i=1}^{n} u_{x_i} \nu^i \, dS = \int_{\partial U} \frac{\partial u}{\partial \nu} \, dS,$$

then taking $v = v_{x_i}$ in the integration by parts formula, we get

$$\int_U u_{x_i} v_{x_i} \, d\boldsymbol{x} = -\int_U u v_{x_i x_i} \, d\boldsymbol{x} + \int_{\partial U} u v_{x_i} \nu^i \, dS$$

Summing, we get

$$\int_U Du \cdot Dv \, d\boldsymbol{x} = -\int_U u \Delta v \, d\boldsymbol{x} + \int_{\partial U} u \frac{\partial v}{\partial \nu} \, dS \tag{10}$$

Interchanging u and v in (10) we can derive a similar formula involving Δv

$$\int_U Du \cdot Dv \, d\boldsymbol{x} = -\int_U v\Delta u \, d\boldsymbol{x} + \int_{\partial U} \frac{\partial u}{\partial \nu} v \, dS \tag{11}$$

Finally, if we subtract (11) and (10) we get

$$\int_U u\Delta v \, d\boldsymbol{x} - \int_{\partial U} \frac{\partial v}{\partial \nu} u \, dS - \int_U v\Delta u \, d\boldsymbol{x} + \int_{\partial U} \frac{\partial u}{\partial \nu} v \, dS = 0 \tag{12}$$

Those relations are simultaneously known as Green's formulas

Theorem 5 (Green's formulas)**.**

$$\int_U \Delta u \, d\boldsymbol{x} = \int_{\partial U} \frac{\partial u}{\partial \nu} \, dS \tag{13}$$

$$\int_U Du \cdot Dv \, d\boldsymbol{x} = -\int_U u\Delta v \, d\boldsymbol{x} + \int_{\partial U} u \frac{\partial v}{\partial \nu} \, dS \tag{14}$$

$$\int_U u\Delta v \, d\boldsymbol{x} - \int_U v\Delta u \, d\boldsymbol{x} = \int_{\partial U} \frac{\partial v}{\partial \nu} u \, dS - \int_{\partial U} \frac{\partial u}{\partial \nu} v \, dS = 0 \tag{15}$$

Motivation and integral formulation

Let u be an arbitrary function in $C^2(\overline{U})$. Let us fix $x \in U$ and take $\varepsilon > 0$ small enough so that $B(x, \varepsilon) \subset U$. Applying (15) to $V_\varepsilon = U \setminus B(x, \varepsilon)$ with $v = \Phi(y - x)$, we obtain

$$\begin{aligned} &\int_{V_\varepsilon} u(\boldsymbol{y})\Delta\Phi(y - x) \, d\boldsymbol{y} - \int_{V_\varepsilon} \Phi(y - x)\Delta u(\boldsymbol{y}) \, d\boldsymbol{y} \\ &= \int_{\partial V_\varepsilon} u(y)\frac{\partial \Phi}{\partial \nu}(y - x) \, dS(y) - \int_{\partial V_\varepsilon} \Phi(y - x)\frac{\partial u}{\partial \nu}(y) \, dS(y) \end{aligned}$$

using $\Delta\Phi(y - x) = 0$ we get

$$-\int_{V_\varepsilon} \Phi(y - x)\Delta u \, d\boldsymbol{y} = \int_{\partial V_\varepsilon} u(\boldsymbol{y})\frac{\partial \Phi}{\partial \nu}(\boldsymbol{y} - \boldsymbol{x}) \, dS(\boldsymbol{y}) - \int_{\partial V_\varepsilon} \Phi(\boldsymbol{y} - \boldsymbol{x})\frac{\partial u}{\partial \nu}(\boldsymbol{y}) \, dS(\boldsymbol{y})$$

On the $\partial B(\boldsymbol{x}, \varepsilon)$ part of the boundary, we have

$$\left| \int_{\partial B(\boldsymbol{x},\varepsilon)} \Phi(\boldsymbol{y} - \boldsymbol{x})\frac{\partial u}{\partial \nu}(\boldsymbol{y}) \, dS(\boldsymbol{y}) \right| \le C\varepsilon^{n-1} \max_{\partial B(0,\varepsilon)} |\Phi(\boldsymbol{y})| \le \begin{cases} C\varepsilon \log|\varepsilon| & n = 2 \\ C\varepsilon & n \ge 3 \end{cases}$$

The constant follows from the fact that since $u \in C^2$ the derivatives are bounded on $\overline{U}$.

Recall that in the proof of Theorem 3, we noted that the derivative of Φ and the outward normal each obeyed

$$D\Phi(\boldsymbol{y}) = -\frac{1}{nV_n}\frac{\boldsymbol{y}}{|\boldsymbol{y}|^n}, \quad \nu = -\frac{\boldsymbol{y}}{|\boldsymbol{y}|} = -\frac{\boldsymbol{y}}{\varepsilon}, \quad \text{on } \partial B(0, \varepsilon)$$

From which

$$\frac{\partial \Phi}{\partial \nu} = \nu \cdot D\Phi(\boldsymbol{y}) = \frac{1}{nV_n\varepsilon^{n-1}} = \frac{1}{|\partial B(x,\varepsilon)|}$$

From this second line we can thus write

$$\int_{\partial V_\varepsilon} u(\boldsymbol{y}) \frac{\partial \Phi}{\partial \nu}(\boldsymbol{y}-\boldsymbol{x})\, dS(\boldsymbol{y}) = ⨍_{\partial B(\boldsymbol{x},\varepsilon)} u(\boldsymbol{y})\, dS(\boldsymbol{y}) \to u(\boldsymbol{x}), \quad \text{when } \varepsilon \to 0$$

All in all, when $\varepsilon \to 0$ we thus recover the integral formulation

$$\begin{aligned} u(\boldsymbol{x}) = &\int_{\partial U} \Phi(\boldsymbol{y}-\boldsymbol{x}) \frac{\partial u}{\partial \nu}(\boldsymbol{y}) - u(\boldsymbol{y}) \frac{\partial \Phi}{\partial \nu}(\boldsymbol{y}-\boldsymbol{x})\, dS(\boldsymbol{y}) \\ &- \int_U \Phi(\boldsymbol{y}-\boldsymbol{x}) \Delta u(\boldsymbol{y})\, d\boldsymbol{y}. \end{aligned} \tag{16}$$

The formulation holds for every $\boldsymbol{x} \in U$ and any function $u \in C^2(\overline{U})$. Recall that we showed that when the integral is understood as an improper integral, and as soon as $f \in C_c^2(\overline{U})$, then $u \in C^2(\overline{U})$. Formula (16) is particularly attractive. Indeed, when looking for a solution to Poisson's equation for example, we know that the Laplacian Δu is given by the source term f on U. Moreover, the value of u is also known on the boundary ∂U by means of the Dirichlet conditions. A difficulty remains however as we don't know the value of $\partial_\nu u$ on ∂U. In order to derive a useful formula for the solution u, we would therefore like to get rid of this first term.

Let us go back to Green's identity

$$\int_U \Delta u\, v - \Delta v\, u\, d\boldsymbol{x} = \int_{\partial U} v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}\, dS \tag{17}$$

From this formula, if we could find a function v such that $v = \Phi$ on ∂U and $\Delta v = 0$ on U, we could then get rid of the $\partial_\nu u$ term in (17). Let $v = \varphi$ denote such a function, i.e. let us assume that φ satisfies $\Delta \varphi = 0$ on U as well as $\varphi = \Phi$ on ∂U. Substituting this function in (17), we get

$$\int_U \Delta u\, \varphi\, d\boldsymbol{x} = \int_{\partial U} \Phi \frac{\partial u}{\partial \nu}\, d\boldsymbol{x} - u \frac{\partial \Phi}{\partial \nu}\, dS$$

which we can then use to get rid of the $\partial_\nu u$ term in (16)

$$\begin{aligned} u(\boldsymbol{x}) = &\int_U \Delta u\, \varphi\, d\boldsymbol{x} + \int_{\partial U} u \frac{\partial \varphi}{\partial \nu}\, dS - \int_U \Phi(\boldsymbol{y}-\boldsymbol{x}) \Delta u(\boldsymbol{y})\, d\boldsymbol{y} \\ &- \int_{\partial U} u \frac{\Phi}{\partial \nu}\, dS \end{aligned}$$

Letting $G(\boldsymbol{x},\boldsymbol{y}) = \Phi(\boldsymbol{y}-\boldsymbol{x}) - \varphi(\boldsymbol{y})$, this last expression can be reduced to

$$u(\boldsymbol{x}) = -\int_U \Delta u \, G(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y} - \int_{\partial U} u \frac{\partial G}{\partial \nu} \, dS \tag{18}$$

Provided that we can compute the function $G(\boldsymbol{x}, \boldsymbol{y})$ (known as the Green function) formula (18) is a particularly efficient formula since, as was noted earlier, it gives us a closed form expression for the solution to any Dirichlet problem (involving Poisson's equation of course).

Deriving the Green function

Note that $\varphi(\boldsymbol{y})$ satisfies $\Delta\varphi = 0$ on all of U and $\varphi(\boldsymbol{y}) = \Phi(\boldsymbol{x} - \boldsymbol{y})$ on ∂U. φ is thus very similar to Φ which satisfies $\Delta\Phi = 0$ everywhere except at 0 where it is not defined. One idea to construct φ is to start from Φ and try to move the singularity of Φ outside of the domain U of $G(\boldsymbol{x}, \boldsymbol{y})$. As an example, let us consider the half space $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3_+$ (note that $\mathbb{R}^3_+ = (x_1, x_2, x_3)$ with $x_3 > 0$). If we take $\tilde{x} = (x_1, x_2, -x_3)$, then the function

$$\varphi^x(\boldsymbol{y}) = \Phi(\boldsymbol{y} - \tilde{\boldsymbol{x}}) = \frac{1}{n(n-2)V_n} \frac{1}{|\boldsymbol{y} - \tilde{\boldsymbol{x}}|^{n-2}}$$

clearly satisfies $\varphi^x(\boldsymbol{y}) = \Phi(\boldsymbol{y} - \boldsymbol{x})$ on the boundary (i.e. $x_3 = 0$). Moreover, for any $\boldsymbol{x}$, the singularity of φ is now located at $\tilde{\boldsymbol{x}}$ and hence moved into the lower half space which leads to $\Delta\varphi^x = 0$ on $\mathbb{R}^3_+$ (since Laplace's equation is translation invariant). We can thus define our Green function as

$$G(\boldsymbol{x}, \boldsymbol{y}) = \Phi(\boldsymbol{y} - \boldsymbol{x}) - \Phi(\boldsymbol{y} - \tilde{\boldsymbol{x}})$$

A similar idea can be applied to the open ball $B^0(0,1) = \{\boldsymbol{x} \mid |\boldsymbol{x}| < R = 1\}$. In this case, we again want to move the singularity from inside $B(0,1)$ to outside this ball. Proceeding as before, we let

$$\varphi^x(\boldsymbol{y}) = \frac{q}{4\pi|\boldsymbol{x}^* - \boldsymbol{y}|}$$

and try to find $\boldsymbol{x}^*$ (outside $B(0,1)$) and q such that for $|\boldsymbol{y}| = R = 1$,

$$\frac{q}{4\pi|\boldsymbol{x}^* - \boldsymbol{y}|} = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}, \qquad \text{(i.e. on } \partial B^0(0,1)) \tag{19}$$

From (19), letting $|\boldsymbol{y}| = 1$ we get

$$\begin{aligned} &|\boldsymbol{x}^* - \boldsymbol{y}|^2 = q^2|\boldsymbol{x} - \boldsymbol{y}|^2 \\ \Leftrightarrow &|\boldsymbol{x}^*|^2 + 1 - 2\langle \boldsymbol{x}^*, \boldsymbol{y}\rangle = q^* \left(|\boldsymbol{x}|^2 + 1 - 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right) \end{aligned}$$

Rearranging, we get

$$|\boldsymbol{x}^*|^2 + 1 - q^2\left(1 + |\boldsymbol{x}|^2\right) = 2\boldsymbol{y} \cdot (\boldsymbol{x}^* - q^2\boldsymbol{x})$$

Since the LHS does not depend on $\boldsymbol{y}$, the only possibility is to have $\boldsymbol{x}^* = q^2\boldsymbol{x}$ which gives

$$q^4|\boldsymbol{x}|^2 + 1 - q^2(1 + |\boldsymbol{x}|^2) = 0$$

from which we get $q = |\boldsymbol{x}|^{-1}$. We can therefore define our Green function as

$$G(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi}\frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} - \frac{1}{4\pi}\frac{1}{|\boldsymbol{x}||\boldsymbol{x}^* - \boldsymbol{y}|}, \quad \text{with } x^* = \frac{1}{|\boldsymbol{x}|^2}\boldsymbol{x}$$

The mapping $\boldsymbol{x} \mapsto \boldsymbol{x}^* = \frac{R^2}{|\boldsymbol{x}|^2}\boldsymbol{x}$ is called inversion through the sphere $\partial B(0, R)$ and $\boldsymbol{x}^*$ is known as dual of $\boldsymbol{x}$ with respect to $\partial B(0, R)$.

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