Partial Differential Equations, lecture 1

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Introduction

Let $u(\mathbf{x}) \quad : \mathbf{x} \in \mathcal{U} \subseteq \mathbb{R}^n \mapsto u(\mathbf{x}) \in \mathbb{R}$. A partial differential equation (PDE) is an equation involving an unknown function $u(x)$ of two or more variables and certain of its partial derivatives. Recall that the partial derivative of a function $u(x)$

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} \tag{1}$$

which we will sometimes write compctaly as u_{x_i}. Similarly we have

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_i x_j}, \quad \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} = u_{x_i x_j x_k}$$

More generally, given a multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ such that $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n$, we will consider the notation

$$D^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u(\mathbf{x}) \tag{2}$$

If k is a non negative integer, the set $\{D^{\boldsymbol{\alpha}} U(\mathbf{x}) || \boldsymbol{\alpha}| = k\}$ will sometimes be represented compactly as $D^k u(\mathbf{x})$ (denoting the set of all partial derivatives of order k). When $k = 2$, the elements of $D^2 u$ can be rearranged as the matrix

$$D^2 u = \begin{pmatrix} u_{x_1 x_1} & \ldots & u_{x_1 x_n} \\ \vdots & & \vdots \\ u_{x_n x_1} & \ldots & u_{x_n x_n} \end{pmatrix}$$

known as the Hessian matrix of $u(\mathbf{x})$.

We will use the notation Δu to denote the Laplacian of $u \in \mathbb{R}^n$,

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \mathrm{div} Du = \nabla \cdot \nabla u = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \cdot \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) \tag{3}$$

$$= \mathrm{tr}\left(D^2 u\right) \tag{4}$$

Now that we have all the necessary tools, we can introduce a more formal defintion of PDEs. We call a k^{th} order partial differential equation, an expression of the form

$$F\left(D^k u(\mathbf{x}), D^{k-1} u(\mathbf{x}, \ldots, Du(\mathbf{x}), u(\mathbf{x}), u(\mathbf{x}), \mathbf{x})\right) = 0, \quad \mathbf{x} \in \mathcal{U} \tag{5}$$

where $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ is an known function an $u(\mathbf{x}) : \mathcal{U} \mapsto \mathbb{R}$.

Solving the PDE (5) corresponds to finding all the solutions $u(\mathbf{x})$ satisfying (5) possibly among all the functions satisfying certain auxilliary boundary conditions on some part Γ of $\partial\mathcal{U}$.

By finding solutions, we mean obtaining simple explicit solutions or at least prove properties of those solutions.

A first important distinction is between linear and non linear equations.

We call a PDE linear if F is linear with respect to $u(\mathbf{x})$ and all its derivatives or similarly if F is of the form

$$F(D^k u, \ldots, u, \mathbf{x}) = \sum_{|\boldsymbol{\alpha}| \leq k} a_{\boldsymbol{\alpha}}(\mathbf{x}) D^{\alpha} u + f(\mathbf{x}) \tag{6}$$

for some given functions $a_{\boldsymbol{\alpha}}(\mathbf{x})$ ($|\boldsymbol{\alpha}| \leq k$). Note that we call the PDE homogeneous if $f(\mathbf{x}) = 0$.

A second distinction concerns the type of non linearity. A general feature of PDEs is that the terms with the highest number of derivatives, called principal terms matter most typically. As a result, when moving to non linear PDEs, it makes a big difference in which terms the non linearity shows up.

- A k^{th} order PDE which is of the form

  $$\sum_{|\boldsymbol{\alpha}| = k} a_{\boldsymbol{\alpha}}(\mathbf{x}) \partial_u^{\boldsymbol{\alpha}} u = F(\mathbf{x}, u, \partial u, \ldots, \partial^{k-1} u) \tag{7}$$

  that is where the non linearity at most enters in all the terms of order at most $k-1$, is called semilinear.

  Another way to look at this is to say that a semilinear PDE is linear in the leading order derivatives. For instance

  $$\Delta u = u^2 + \sum_{j=1}^{n} (\partial_j u)^2 + u^2 \tag{8}$$

  is a semilinear equation.

- If one allows the coefficients $a_{\boldsymbol{\alpha}}$ on the left-hand side to depend on u as well as its derivatives up to order $k-1$,

$$\sum_{|\boldsymbol{\alpha}|=k} a_\alpha(\mathbf{x}, u, \partial u, \ldots, \partial^{k-1}u)\,(\partial^\alpha u)\,(\mathbf{x}) = F(\mathbf{x}, u, \partial u, \ldots, \partial^{k-1}u) \tag{9}$$

  the equation is called quasilinear. An example is

$$(1+u_x^2)u_{xx} + (1+u_y^2)u_{yy} = f(x,y) \tag{10}$$

- Finally, the PDE is fully non-linear, if it depends non linearly on the highest order derivatives.

The theory of linear equations can be considered sufficiently well developed an consolidated, at least for what concerns the most important questions. On the contrary, the non-linearities present such a rich variety of aspects and complications that a general theory does not appear to be conceivable. Existing results and the new investigations focus on more or less specific cases, especially those that are interesting in the applied sciences.

let us now look at a couple of important linear and non linear equations:

1. Transport Equations (first order)

$$u_t + \boldsymbol{v} \cdot \nabla u = 0 \tag{11}$$

   The transport equation describes for instance, the transport of a solid poluting substance along a channel. Here u is the concentration of the substance and $\boldsymbol{v}$ is the stream speed.

2. Diffusion or Heat equation. (second order)

$$u_t - D\Delta u = 0 \tag{12}$$

   Where $\Delta = \partial_{x_1 x_1} + \ldots + \partial_{x_n x_n}$ is the Laplace operator. The diffusion equation describes the conductionof heat through a homogeneous and isotropic medium; u is the temperature and D encodes the thermal properties of the material.

3. Wave equation (second order)

$$u_{tt} - c^2\Delta u = 0 \tag{13}$$

   describes for instance the propagation of transversal waves of small amplitude in a perfectly elastic chord (e.g. of a violin) if $n = 1$, or membrane (e.g. of a drum) if $n = 2$. if $n = 3$, it can be used to describe the propagation of electromagnetic waves in a vacuum or of small amplitude sound waves. In those frameworks, u may represent the wave amplitude and c denotes the propagation speed.

4. Laplace's or potential equation (second order)

$$\Delta u = 0 \tag{14}$$

where $u = u(\mathbf{x})$. The diffusion and the wave equations model evolutions phenomena. The Laplace equation describes the corresponding steady state in which the solution does not depend on time anymore. Together with its non homogeneous version

$$\Delta u = f \tag{15}$$

called Poisson's equation, it also plays an important role in electrostatics

5. Black-Scholes equation (second order)

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0 \tag{16}$$

Here $u = u(x,t)$, $x \geq 0$, $t \geq 0$. Fundamental in mathematical finance, this equation governs the evolution of the price u of a derivatie (European option) based on some underlying asset (a stock or a currency), whose price is x.

6. Vibrating plate (fourth order)

$$u_{tt} - \Delta^2 u = 0 \tag{17}$$

where $\mathbf{x} \in \mathbb{R}^2$ and

$$\Delta^2 u = \Delta(\Delta u) = \frac{\partial^4 u}{\partial x_1^4} + 2\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u}{\partial x_2^4} \tag{18}$$

is called the *biharmonic operator*. In the theory of linear elasticity, it models the transversal waves of small amplitudes of a homogeneous isotropic plate.

7. Schrödinger equation (second order)

$$-iu_t = \Delta u + V(x)u \tag{19}$$

where i is the complex unit. This equation is fundamental in quantum mechanics and governs the evolution of a particle subject to a potential V. The function $|u|^2$ represents a probability density.

Among the most popular non linear PDEs we should mention

1. Burgers equation (quasi-linear, first order)

$$u_t + cuu_x = 0, \quad (x \in \mathbb{R}) \tag{20}$$

Burger's equation governs the one dimensional flux of a non viscous fluid but it can be used to model traffic dynamics as well. Its viscous variant

$$u_t + cuu_x = \varepsilon u_{xx} \tag{21}$$

constitutes a basic example of competition between dissipation (due to the term εu_{xx}) and steepening (i.e. shock formation due to the term cuu_x)

2. Fisher's equation (semilinear, second order)

$$u_t - \Delta u = ru(M - u) \tag{22}$$

(D, r, M are positive constants). Fisher's equation governs the evolution of a population of density u subject to diffusion and logistic growth (represented by the right-hand side)

3. Porous medium equation (quasi-linear, second order)

$$u_t = k\text{div}\,(u^{\gamma}\nabla u) \tag{23}$$

where $k > 0$, $\gamma > 1$ are constants. This equation appears in the description of the filtration phenomena, e.g. motion of water through the ground.

4. Minimal surface equation (quasi-linear, second order)

$$\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0, \quad (x \in \mathbb{R}^2) \tag{24}$$

The graph of a solution u minimizes the area under all the surfaces $z = v(x_1, x_2)$ whose boundary is a given curve. For example, soap bubbles are minimal surfaces.

5. Eikonal equation (fully non-linear, first order)

$$|\nabla u| = c(x) \tag{25}$$

The eikonal equation appears in geometrical optics. if u is a solution, its level surfaces $u(\mathbf{x} = t)$ describe the position of a light wave front at time t.

Finally we describe a few important systems of equtions

1. Navier's equation for linear elasticity (three scalar equations of second order)

$$\rho \mathbf{u}_{tt} = \mu \Delta \mathbf{u} + (\mu + \lambda)\ \text{grad div}\ \mathbf{u} \tag{26}$$

where $\mathbf{u} = (u_1(x,t)u_2(x,t), u_3(x,t))$, $\mathbf{x} \in \mathbb{R}^3$. The vector $\mathbf{u}$ represents the displacement from the equilibrium of a deformable continuum body of (constant) density ρ.

2. Maxwell's equations in vacuum (six scalar linear equations of first order)

$$\mathbf{E}_t - \text{curl}\ \mathbf{B} = \mathbf{0}, \quad \mathbf{B}_t + \text{curl}\ \mathbf{E} = \mathbf{0}, \quad \text{(Ampère and Faraday's laws)} \tag{27}$$

$$\text{div}\ \mathbf{E} = \mathbf{0}, \quad \text{div}\ \mathbf{B} = \mathbf{0}, \quad \text{(Gauss laws)} \tag{28}$$

Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic induction field , the light speed is $c = 1$ and the magnetic permeability is $\mu_0 = 1$

3. Navier Stokes equations (three quasi-linear scalar equations of second order and one linear equation of first order)

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \Delta u \tag{29}$$

$$\text{div}\ \mathbf{u} = 0 \tag{30}$$

where $\mathbf{u} = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$, $p = p(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^3$. This equation governs the motion of a viscous, homogeneous and incompressible fluid. Here $\mathbf{u}$ is the fluid speed, p is its pressure, ρ its density (constant) abd ν is the kinematic viscosity, given by the ratio between the fluid viscosity and its density. The term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ represents the inertial acceleration due to fluid transport.

In the construction of a mathematical model, only some of the general laws of continuum mechanics are relevant, while the others are eliminated through the constitutive laws or suitably simplified according tot he current situation. In general additional information is necessary to select or to predict the existence of a unique solution. Such additional information is commonly supplied in the form of initial and/or boundary data although other forms are possible. For instance, typical boundary conditions prescribe the value of the solution or of its normal derivative or a combination of the two at the boundary of the relevant domain.

A main goal of a theory is to establish conditions on the data in order to have a problem with the following features:

(i) There exists at least one solution

(ii) There exists at most one solution

(iii) The solution depends continuously on the data

The last condition requires some explanation. Roughly speaking, property (iii) states that the correspondence data $\to$ solution is continuous or, in other words, that a small error on the data entails a small error on the solution. This property extremely important and may be expressed as a local stability of the solution with respect to the data. Think for instance of using a computer to find an approximate solution: the insertion of the data an the computation algorithms entail approximation errors of various types. A significant sensitivity of the solution on small variations of the data would produce an unacceptable result.

The notion of continuity and the error measurements, both in the data and in the solution are made precise by introducing a suitable notion of distance. When dealing with a numerical or a finite dimensional set of data, an appropriate distance may be the Euclidean distance. if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$

$$\text{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} \tag{31}$$

When dealing with real functions defined on a set A, common distances are

$$\text{dist}(f, g) = \max_{x \in A} |f(x) - g(x)| \tag{32}$$

which measures the maximum difference between f and g over A or

$$\text{dist}(f, g) = \sqrt{\int_A (f - g)^2} \tag{33}$$

which is the L^2 distance between the function f and g. When a problem possesses the properties (i), (ii) and (iii) above, it is said to be well-posed.

When using a mathematical model, it is extremely useful, sometimes essential to deal with well posed problems. Uniqueness and stability increase the possibility of providing accurate numerical approximations.

On the other hand, well posed problems are not the only problems that are interesting. There exist problems that are intrinsically ill-posed because of the lack of uniqueness or of stability but are still of great interest for practical applications. An important class of ill posed problems is the class of inverse problems.

Another important question as we will see is "What do we mean by the term 'solution'"? Should we require our 'solution' to be real analytic? or infinitely differentiable?

Perhaps this is too much and we should only require a solution to an order k PDE to be k times continuously differentiable. This will ensure that the derivatives which appear in the statement of the PDE will exist and be continuous. We will call a solution with such smoothness a classical solution. By solving a PDE in the classical sense, we will then mean writing down a formula for a classical solution satisfying the existence, uniqueness and well-posedness conditions.

Solving a PDE in the classical sense is however not always possible. If we consider the scalar conservation law for example

$$u_t + F(u)_x = 0 \tag{34}$$

which is used to describe the formation and propagation of shock waves. A shock wave is a curve of discontinuity so if we want to study conservation laws, we must therefore allow for solutions u which are not continuously differentiable or even continuous.

In general the conservation law has no classical solution. It is however well posed if we allow for properly defined generalized or weak solutions. The search for generalized solutions can in fact be applied more generally, including to problems which admit classical solutions.

A good strategy is to consider as separate the existence problem and the smoothness problem. For any given PDE, we can define at first a relatively wide notion of weak solution with the hope that in this framework, it might be easier to establish existence, uniqueness and continuous dependence on the data.

For some PDEs, deriving a weak solution might be the best we can achieve. For others, it might turn out that our weak solutions are ultimately smooth enough to qualify as classical solutions. This idea is encoded in the notion of regularity.

References