Laplace + Poisson's equations

- Harmonic functions
- Mean Value formulas

Today: - Maximum Principle (Harnack's inequality)

→ Fundamental Solution

→ Solution of Poisson's equation

→ Green function

Theorem (Harnack's inequality)

For each connected open set $V \subset\subset U$ (meaning $V \subset \bar{V} \subset U$ and $\bar{V}$ compact), there is a positive constant C depending only on V

$$\sup_V u \leq C \inf_V u$$

for all non negative functions u in U

More particularly we have

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y)$$

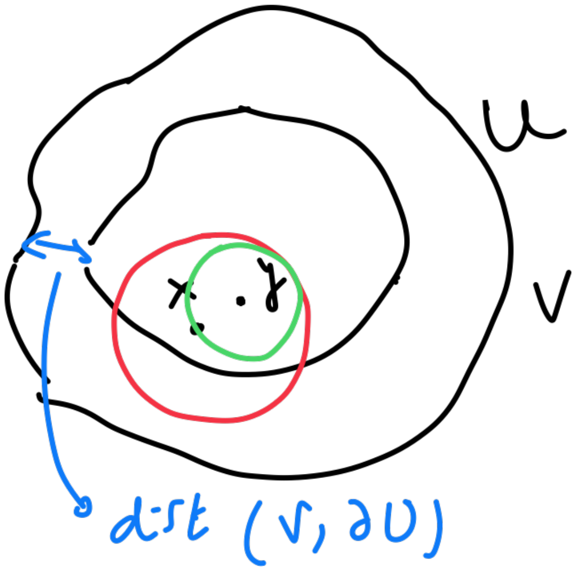
for every $x, y \in V$

Proof $\quad r = \frac{1}{4}\,\mathrm{dist}(V, \partial U) \qquad x, y \in V \qquad |x-y| \le r$

V_n = volume of radius 1 ball in $\mathbb{R}^n$

Mean Value Formula

$$u(x) = \fint_{B(x,2r)} u \, dz \ge \frac{1}{V_n 2^n r^n} \int_{B(y,r)} u \, dz$$



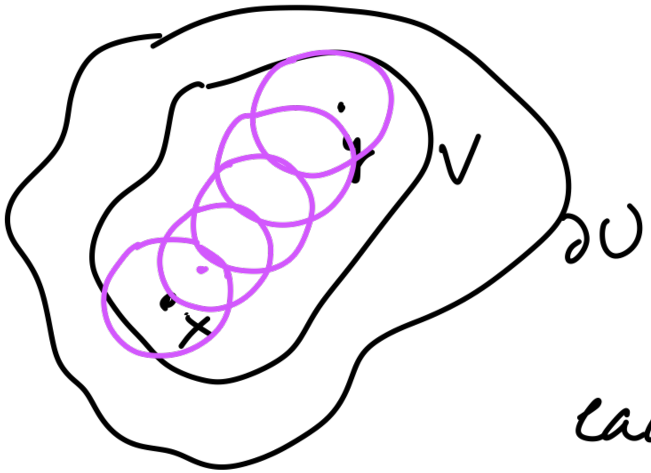
$$\ge \frac{1}{2^n} \fint_{B(y,r)} u \, dz \ge \frac{1}{2^n} u(y)$$

$$u(x) \ge \frac{1}{2^n} u(y)$$

Since x, y arbitrary, we can swap them hence

$$\Rightarrow \quad u(y) \ge \frac{1}{2^n} u(x)$$

$$\Rightarrow \frac{1}{2^n} u(y) \le u(x) \le 2^n u(y)$$



Since V is connected and $\bar{V}$ is compact $\Rightarrow$ we can cover $\bar{V}$ with a finite chain of balls $\{B_i\}_{i=1}^N$ each of radius r and with $B_i \cap B_{i-1} \neq \emptyset$

so that

$$u(x) \ge \frac{1}{2^{n(N+1)}} u(y)$$

$\square$

Fundamental Solution

invariants

- translations : if $u(x)$ is harmonic, so is $u(x-y)$ for any fixed y

- Rotation in $\mathbb{R}^n$ we represent a rotation by means of an orthogonal matrix

M s.t $(M^{-1} = M^T)$

if $u(x)$ is harmonic, so is $v(x) = u(Mx)$

To check this $\quad \Delta u = Tr(D^2 u)$

$$D^2 u = \begin{bmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & & \\ \frac{\partial^2 u}{\partial x_1 \partial x_2} & \frac{\partial^2 u}{\partial x_2^2} & & \\ & & \ddots & \\ & & & \frac{\partial^2 u}{\partial x_n^2} \end{bmatrix}$$

$$D^2 v = D^2(Mx) = M^T D^2 u(Mx) \, M \quad (\text{exercise})$$

Then $\Delta v = \mathrm{Tr}\left(M^T D^2 u(Mx) \, M \right)$

$$= \mathrm{Tr}\left(D^2 u(Mx) \right)$$

$$= \Delta u = 0$$

$\to$ General idea : How about searching for solutions of the form $u(|x|) = u(r)$? that only depend on the radius

$$|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$\to$ $d=2$ in cylindrical coordinates

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0$$

$$\frac{\frac{\partial^2 u}{\partial r^2}}{\frac{\partial u}{\partial r}} = -\frac{1}{r} \to \log \frac{\partial u}{\partial r} = -\log r + C$$

$$\frac{\partial u}{\partial r} = \frac{C}{r}$$

$$u(r) = C \log r + C'$$

→ $d=3$ → in spherical coordinates the laplace operator is given by

$$\Delta = \underbrace{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}}_{\text{radial part}} + \frac{1}{r^2}\Big\{\underbrace{\frac{1}{(\sin\psi)^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\psi^2} + \cot\psi\frac{\partial}{\partial\psi}}_{\text{laplace Beltrami operator}}\Big\}$$

We restrict ourselves to the radial part

$$\frac{\partial^2 u}{\partial r^2} \Big/ \frac{\partial u}{\partial r} = -\frac{2}{r}$$

integrating both sides we get $\log\frac{\partial u}{\partial r} = -2\log(r) + C$

$$\frac{\partial u}{\partial r} = \frac{c}{r^2} \rightarrow \boxed{u(r) = -\frac{c}{r} + c'}$$

the Fundamental solution of Laplace's equation is defined as

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & n=2 \\ \frac{1}{4\pi|x|} & n=3 \end{cases}$$

More generally

$$\Phi(x) = \frac{1}{n(n-2)V_n} \frac{1}{|x|^{n-2}} \quad n>3$$

the constants c, c' are set for u to satisfy

$$\boxed{\Delta u = \delta_n(x)}$$

From the Fundamental Solution, we can build solutions to Poisson's equation $\Delta u = f$

through the Newtonian potential

Suppose $(4\pi)^{-1} f(x)$ represents the density of charge inside a compact set in $\mathbb{R}^3$

Then $\Phi(x-y) f(y) dy$ is the contribution to the potential of a small charge $(4\pi)^{-2} f(y) dy$ located inside the small volume dy

The full potential u is the sum of all contributions

$$u(x) = \int_{\mathbb{R}^3} \Phi(x-y) f(y) \, dy = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} \, dy$$

$u(x)$ is the convolution between $f(y)$ and the fundamental solution $\Phi(x-y)$ and is known as the Newtonian potential.

In particular

$$\Delta u = \int_{\mathbb{R}^3} \Delta_x \Phi(x-y)\, f(y)\, dy = \int_{\mathbb{R}} \delta(x-y)\, f(y)\, dy = f(x)$$

Theorem (Solution of Poisson's equation)

Let $f \in C^2(\mathbb{R}^3)$ with compact support. Let $u(x)$ be defined as the Newtonian potential, i.e

$$u(x) = \int_{\mathbb{R}^3} \Phi(x-y) f(y)\, dy$$

Then u is the only solution in $\mathbb{R}^3$ of $\Delta u = -f$ belonging to $C^2(\mathbb{R}^3)$ and vanishes at ∞.

Step 1 Uniqueness

let us assume $u, v \in C^2(\mathbb{R}^3)$ both solutions $\Delta u = \Delta v = -f$

and both vanishing at ∞

using our Maximum principle

$$\sup (u-v) \leq C \inf (u-v)$$

since u, v vanish at ∞ $\Rightarrow \inf (u-v) = \inf (v-u) = 0$

$$\Rightarrow \max(u-v) \leq 0 \quad \max (v-u) \leq 0$$

$$\max |u-v| \leq 0 \Rightarrow u = v$$