Fundamental Solution

- invariant transformations

(translation) $u(x,t)$ is a solution, so is $u(x-s, t-\tau)$
$s, \tau \in \mathbb{R}$

(dilations) $u(x,t)$ is a solution, so is

$c\, u(ax, a^2 t)$ for any $a, c \in \mathbb{R}$

Often we are interested in solutions satisfying the conservation of energy

heat energy $e(x,t) = c \cdot \rho\, u$

$$\int_{\mathbb{R}} u(x,t)\, dx = q \in \mathbb{R}$$

Question: for what value of a, c does $c\,u(ax, a^2t)$ satisfies conservation of energy?

$$1 = \int_{\mathbb{R}^n} u(x,t)\,dx \quad \text{for all } t$$

$$\rightarrow \int_{\mathbb{R}^n} c\,u(ax, a^2t)\,dx = 1 = \int_{\mathbb{R}^n} \frac{c}{a^n} u(y, a^2t)\,dy$$

$$\rightarrow \frac{c}{a^n} = 1$$

$\Rightarrow$ energy conserved if $\frac{c}{a^n} = 1$

$$\frac{|x|^2}{\Delta t} \quad \frac{|x|}{\sqrt{\Delta t}} \rightarrow \text{unaffected by parabolic dilations}$$

to find a solution → look for function $U\left(\frac{|x|}{\sqrt{\Delta t}}\right)$

+ conservation of heat energy

$$\int_{\mathbb{R}^n} U\left(\frac{|x|}{\sqrt{\Delta t}}\right) dx = 1 \quad \text{at every } t$$

$$\frac{1}{\lambda}\sqrt{x_1^2 + x_2^2 + \ldots} = \sqrt{\left(\frac{x_1}{\lambda}\right)^2 + \left(\frac{x_2}{\lambda}\right)^2 + \ldots}$$

⇒ in particular we must have $\int_{\mathbb{R}^n} U\left(\frac{|x|}{\sqrt{\Delta t}\,\lambda}\right) dx = 1 \quad \forall \lambda$

$$\int_{\mathbb{R}^n} U\left(\frac{\frac{|x|}{\lambda}}{\sqrt{\Delta t}}\right) dx = \int_{\mathbb{R}^n} U\left(\frac{|y|}{\sqrt{\Delta t}}\right) \underbrace{\lambda^n}\, dx$$

To satisfy conservation of energy, it is better to take

$$\frac{1}{(Dt)^{n/2}} U\left(\frac{|x|}{\sqrt{Dt}}\right) = u(x,t)$$

$$u_t - D\Delta u = 0$$

$$u_t = \frac{1}{(D)^{n/2}}\left(-\frac{n}{2} t^{-n/2-1} U\left(\frac{|x|}{\sqrt{Dt}}\right) + \frac{1}{t^{n/2}} U'\left(\frac{|x|}{\sqrt{Dt}}\right) \cdot \frac{|x|}{\sqrt{D}} \cdot -\frac{1}{2} t^{-1/2-1}$$

let $\xi = \dfrac{|x|}{\sqrt{Dt}}$ $\rightarrow$ $u_t = \dfrac{1}{(Dt)^{n/2}}\left(-\dfrac{n}{2t} U(\xi) + \xi U'(\xi) \, -\dfrac{1}{2}\dfrac{1}{t}\right)$

$$= -\frac{1}{2t}\frac{1}{(Dt)^{n/2}}\left(U(\xi) + \xi U'(\xi)\right)$$

$$u_{x_i} = \frac{1}{(Dt)^{n/2}} U'\left(\frac{|x|}{\sqrt{Dt}}\right) \cdot \frac{x_i}{\sqrt{Dt}\,|x|}$$

$$u_{x_i x_i} = \frac{1}{(Dt)^{n/2}} U''\left(\frac{|x|}{\sqrt{Dt}}\right) \cdot \frac{x_i^2}{Dt\,|x|^2} + \frac{1}{(Dt)^{n/2}} U'\left(\frac{|x|}{\sqrt{Dt}}\right) \times \left(\frac{1}{\sqrt{Dt}\,|x|} - \frac{x_i^2}{\sqrt{Dt}\,|x|^3}\right)$$

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \frac{1}{(Dt)^{n/2}} \left(U''(\xi)\,\frac{1}{Dt} + U'(\xi)\left(\frac{n\xi}{|x|} - \frac{\xi}{|x|}\right)\right)$$

$$-\frac{1}{2\Delta t}\left(U(\xi)+\xi U'(\xi)\right)-\left(U''(\xi)\frac{1}{\Delta t}+U'(\xi)\left(\frac{n\xi}{|x|}-\frac{\xi}{|x|}\right)\right)=0$$

multiply by $|x|$

$$\to -\frac{1}{2}\xi\left(U(\xi)+\xi U'(\xi)\right)-\left(U''(\xi)\xi+U'(\xi)(n-1)\xi\right)=0$$

$$u^*(x,t) = \frac{1}{(Dt)^{n/2}}\, U\left(\frac{|x|}{\sqrt{Dt}}\right)$$

→ substitute into $u_t - D\Delta u = 0$

$$u_t = -\frac{n}{2}\frac{1}{(D^{n/2})}\, t^{-n/2-1}\, U\left(\frac{|x|}{\sqrt{Dt}}\right) + \frac{1}{(Dt)^{n/2}}\, U'\left(\frac{|x|}{\sqrt{Dt}}\right)\cdot \frac{|x|}{D^{1/2}}\left(-\frac{1}{2}\right) t^{-3/2}$$

$$u_{x_i} = \frac{1}{(Dt)^{n/2}}\, \frac{1}{2}\, \frac{2x_i}{\sqrt{Dt}}\, \frac{1}{|x|}\, U'\left(\frac{|x|}{\sqrt{Dt}}\right)$$

$$u_{x_i x_i} = \frac{1}{(Dt)^{n/2}}\left(\frac{1}{|x|\sqrt{Dt}} - \frac{x_i^2}{\sqrt{Dt}}\cdot\frac{1}{|x|^3}\right) U'\left(\overbrace{\frac{|x|}{\sqrt{Dt}}}^{=\,\xi}\right)$$

$$+ \frac{1}{(Dt)^{n/2}}\, \frac{x_i}{\sqrt{Dt}}\, \frac{1}{|x|}\, U''\left(\frac{|x|}{\sqrt{Dt}}\right)\cdot\frac{1}{\sqrt{Dt}}\cdot\frac{x_i}{|x|}$$

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \frac{1}{(Dt)^{n/2}} \left(\frac{n}{|x|\sqrt{Dt}} - \frac{|x|^2}{\sqrt{Dt}} \frac{1}{|x|^3} \right) U'(\xi)$$

$|x|$

$$+ \frac{1}{(Dt)^{n/2}} \frac{|x|^2}{(Dt)} \frac{1}{|x|^2} U''(\xi)$$

ξ

$$u_t - D\Delta u = -\frac{n}{2} \frac{1}{(D)^{n/2}} t^{-n/2-1} U(\xi) - \frac{1}{2} \frac{1}{(Dt)^{n/2}} U'(\xi) \frac{|x|}{(Dt)^{1/2}} \frac{1}{t}$$

$$+ \frac{1}{(Dt)^{n/2}} \left(\frac{n-1}{|x|\sqrt{Dt}} \right) U'(\xi) + \frac{1}{(Dt)^{n/2+2}} U''(\xi)$$

$$0 = -\frac{n}{2} \frac{1}{(Dt)^{n/2}} \frac{1}{t} U(\xi) - \frac{1}{2} \frac{1}{(Dt)^{n/2}} U'(\xi) \xi / t$$

$$+ D \frac{1}{(Dt)^{n/2}} \frac{n-1}{|x|\sqrt{Dt}} U'(\xi) + D \frac{1}{(Dt)^{n/2}} \frac{1}{Dt} U''(\xi)$$

$$= -\frac{n}{2}\frac{1}{t}U(\xi) - \frac{1}{2}\frac{1}{t}\xi U'(\xi)$$

$$+ \frac{D\,(n-1)}{|x|\sqrt{Dt}} U'(\xi) + \frac{1}{t}U''(\xi) = 0$$

$$= -\frac{n}{2}U(\xi) - \frac{\xi}{2}U'(\xi) + \frac{\sqrt{Dt}}{|x|}(n-1)U'(\xi) + U''(\xi)$$

$$= -\frac{n}{2}U(\xi) - \frac{\xi}{2}U'(\xi) + \frac{1}{\xi}(n-1)U'(\xi) + U''(\xi) = 0$$

$$= -\frac{n}{2}\xi U(\xi) - \frac{\xi^2}{2}U'(\xi) + (n-1)U'(\xi) + \xi U''(\xi) = 0$$

Multiply by ξ^{n-2}

$$-\frac{n\,\xi^{n-1}}{2}U(\xi) - \frac{\xi^n}{2}U'(\xi) + (n-1)\xi^{(n-2)}U'(\xi)$$

$$+\,\xi^{n-1}U''(\xi)$$

$$\Rightarrow \left(-\frac{\xi^n}{2}U(\xi)\right)' + \left(\xi^{n-1}U'(\xi)\right)' = 0$$

$$-\frac{\xi^n}{2}U(\xi) + \xi^{n-1}U'(\xi) = C$$

$$\lim_{\xi\to 0} U(\xi) < \infty \Rightarrow C = 0$$

$$\lim_{\xi\to 0}\left(-\frac{\xi^n}{2}U(\xi) + \xi^{n-1}U'(\xi)\right) = 0$$

$$-\frac{\xi^n}{2} U(\xi) + \xi^{n-1} U'(\xi) = 0$$

$$-\frac{\xi}{2} U(\xi) + U'(\xi) = 0$$

$$U'(\xi) = \frac{\xi}{2} U(\xi)$$

$$\int \frac{U'(\xi)}{U(\xi)} = \int \frac{\xi}{2} \longrightarrow \log(U(\xi)) = \frac{\xi^2}{4} + A$$

$$U(\xi) = A \exp\left(-\frac{\xi^2}{4}\right)$$

$$\int_{-\infty}^{\infty} \frac{1}{(Dt)^{n/2}} U\left(\frac{|x|}{\sqrt{Dt}}\right) dx = 1 \quad \forall t$$

conservation of energy

$$\int_{-\infty}^{\infty} \frac{A}{(Dt)^{n/2}} \exp\left(-\frac{|x|^2}{4Dt}\right) dx = 1 \quad \forall t \quad **$$

recall that for any μ, Σ

$$\frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \int \exp\left(-(x-\mu)^T \Sigma^{-1} (x-\mu)\right) dx = 1$$

in particular $\Sigma = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} 2Dt \quad \mu = 0$

$$\frac{1}{(2\pi)^{n/2}(2Dt)^{n/2}}\int_{-\infty}^{\infty} \exp\left(-\frac{|x|^2}{2.2Dt}\right) dx = 1$$

$$\int_{-\infty}^{\infty} \exp\left(-\frac{|x|^2}{4Dt}\right) dx = (2\pi)^{n/2}(2Dt)^{n/2}$$

Substitute this in to the conservation of mass,

$$\frac{A}{(Dt)^{n/2}} \cdot (2\pi)^{n/2}(2Dt)^{n/2} = 1$$

$$A = \frac{1}{(2\pi)^{n/2}(2)^{n/2}} = \frac{1}{(4\pi)^{n/2}}$$

The Fundamental solution for the heat equation

$$\Phi(x,t) = \frac{1}{(4\pi D t)^{3/2}} \exp\left(-\frac{|x|^2}{Dt}\right)$$