Today: Cauchy → existence

→ Duhamel's Method

→ uniqueness for Cauchy problem

Solution for Cauchy Problem

$$\begin{cases} u_t - D u_{xx} = 0 \\ u(x,0) = g(x) \end{cases}$$

$$u(x,t) = \int \Phi(x-y,t) \cdot g(y)\, dy$$

Theorem Assume there exist constant (positive numbers) c and a

$$|g(x)| \le c e^{ax^2}$$

define $u(x,t)$ as

$$u(x,t) = \frac{1}{\sqrt{4\pi D t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4Dt}} g(y)\, dy$$

let $T < \frac{1}{4aD}$ then

(i) there exist positive numbers c_1, A such that

$$|u(x,t)| \le c e^{Ax^2} \quad \text{for all } (x,t) \in \mathbb{R} \times (0,T]$$

(ii) $u \in C^\infty(\mathbb{R}\times(0,T])$ and in the strip $\mathbb{R}\times(0,T]$

$$u_t - Du_{xx} = 0$$

(iii) let $(x,t)\to(x_0,0^+)$, if g is if $g(x)$ is continuous,

$$u(x,t)\to g(x_0)$$

Proof i.e since we assume $T < \frac{1}{4aD}$ $t < \frac{1}{4aD}$

$\rightarrow \exists \, \varepsilon$ s.t $\frac{1}{4DT} - a > \varepsilon$ $\frac{1}{4Dt} - a > \varepsilon$

$\exists \, \varepsilon'$ s.t $\frac{1}{4DT} - a > \frac{\varepsilon'}{4DT}$ $\frac{1}{4Dt} - a > \frac{\varepsilon'}{4Dt}$

$\exists \, \varepsilon, \varepsilon'$ s.t $\frac{1}{4Dt} - a > \varepsilon$ $\forall t$

$\frac{1}{4Dt} - a > \frac{\varepsilon'}{4Dt}$ $\forall t$

(i) We want to show $|u(x,t)| \leq C e^{Ax^2}$

$$|u(x,t)| = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} g(y)\, dy$$

$z = x-y$

$$= \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4Dt}} g(x-z)\, dz$$

$$\leq \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4Dt}} C e^{a(x-z)^2}\, dz \quad \times \quad (*)$$

$$-\frac{z^2}{4Dt} + a(x-z)^2 = ax^2 + \frac{a^2}{\left(\frac{1}{4Dt}-a\right)}x^2$$

$$- \left(\sqrt{-a+\frac{1}{4Dt}}\,z - \frac{a\,x}{\sqrt{\frac{1}{4Dt}-a}}\right)^2$$

let us substitute this in (*)

$$|u(x,t)| \le \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{\left(a+\frac{a^2}{\underbrace{\left(\frac{1}{4Dt}-a\right)}_{>\varepsilon}}\right)x^2} e^{-\left(\sqrt{-a+\frac{1}{4Dt}}\,z - \frac{ax}{\sqrt{\frac{1}{4Dt}-a}}\right)^2} dz$$

$$\le \frac{C e^{\left(a+\frac{a^2}{\varepsilon}\right)x^2}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{-a+\frac{1}{4Dt}}\,z - \frac{ax}{\sqrt{\frac{1}{4Dt}-a}}\right)^2} dz$$

use $$\frac{1}{4Dt} - a > \frac{\varepsilon'}{4Dt}$$

$$\underbrace{\sqrt{\frac{1}{4Dt} - a}} \, \frac{1}{\sqrt{\varepsilon'}} > \frac{1}{\sqrt{4Dt}}$$

From this we get

$$|u(x,t)| \leq C e^{\left(a + \frac{a^2}{\varepsilon}\right)x^2} \cdot \sqrt{\frac{1}{4Dt} - a} \, \frac{1}{\sqrt{\varepsilon'}} \times \int_{-\infty}^{\infty} e^{-\left(\underbrace{\sqrt{\frac{1}{4Dt}-a}\, z}_{z' = \sqrt{\frac{1}{4Dt}-a}\, z} - \frac{ax}{\sqrt{\frac{1}{4Dt}-a}}\right)^2} dz$$

$$|u(x,t)| \leq \frac{C e^{\left(a + \frac{a^2}{\varepsilon}\right)x^2}}{\sqrt{\varepsilon'}} \int_{-\infty}^{\infty} e^{-(z' - \mu)^2} dz' \Big)$$

To conclude the proof of (i) just use

$$\int_{-\infty}^{\infty} e^{-(z'-\mu)^2} dz' = \sqrt{2\pi}$$

For point (ii) we are going to use the following result which enables us to move the derivatives inside the integral

Theorem 1 Suppose $f : X \times [a,b] \to \mathbb{C}$ $(-\infty < a < b < \infty)$ and $f(\cdot, t)$ is integrable for each $t \in [a,b]$ let $F(t) = \int_X f(x,t)\, dx$. Suppose $\frac{\partial f}{\partial t}$ exists and if

$g \in L^1(\mu)$ such that $\left| \frac{\partial f}{\partial t}(x,t) \right| \leq g(x)$ for all (x,t)

then $\underline{F \text{ is differentiable}}$ and $F'(t) = \int_X \frac{\partial f}{\partial t}(x,t)\,dx$

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\Delta t}} e^{-\frac{(x-y)^2}{4\Delta t}} g(y)\,dy$$

$$\partial_t^\alpha \partial_x^\beta \frac{1}{\sqrt{4\pi\Delta t}} e^{-\frac{(x-y)^2}{4\Delta t}} \qquad \partial_t^\alpha \partial_x^\beta \; t^{-\alpha} e^{-\frac{(x-y)^2}{t}}$$

has the form of a sum of terms $t^{-r}(x-y)^s e^{-\frac{(x-y)^2}{4\Delta t}}$

Multiplying by $g(y) \leq C e^{ay^2}$

$$t^{-n}|x-y|^s e^{-\frac{(x-y)^2}{4\Delta t}} e^{ay^2}$$

$$\leq t_0^{-n}(|x|+|y|)^s e^{-\frac{x^2}{4\Delta t}} e^{\frac{-y^2}{4\Delta t}} e^{ay^2} e^{\frac{2xy}{4\Delta t}}$$

use $2xy \leq \frac{1}{b}x^2 + by^2$

(to convince yourself

$(x-by)^2 = x^2 + b^2y^2 - 2byx \geq 0$

$\frac{1}{b}x^2 + by^2 \geq 2xy$

$$\leq t_0^{-n}\underbrace{(|x|+|y|)^s}_{\text{see below}} e^{\left(-\frac{1}{4\Delta t}+\frac{1}{b}\right)x^2} e^{-\left(\frac{1}{4\Delta t}-a-\frac{1}{4\Delta t}b\right)y^2}$$

> 0 provided b small enough

$> \varepsilon$ by assumption

$$(|x|+|y|) \leq 2\max(|x|,|y|)$$

hence $(|x|+|y|)^s \leq 2^s|x|^s + 2^s|y|^s$

$$\leq t_0^{-n} 2^s |x|^s e^{\left(-\frac{1}{4\Delta t}+\frac{1}{b}\right)x^2} \quad e^{-\varepsilon y^2} \leftarrow$$

$$+ t_0^{-n} 2^s |y|^s e^{\left(-\frac{1}{4\Delta t}+\frac{1}{b}\right)x^2} \quad e^{-\varepsilon y^2} \leftarrow$$

$$e^{|y|} = \sum_{k=0}^{\infty} \frac{|y|^k}{k!} \geq \frac{|y|^s}{s!}$$

From this, $|y|^s \leq s!\, e^{|y|}$

$$t_0^{-n}\, 2^s\, |y|^s\, e^{\left(-\frac{1}{4Dt}+\frac{1}{b}\right)x^2}\, e^{-\varepsilon y^2}$$

$$\leq t_0^{-2}\, 2^s\, e^{\left(-\frac{1}{4Dt}+\frac{1}{b}\right)x^2}\, s!\, \underbrace{e^{|y|}\, e^{-\varepsilon y^2}}_{\underbrace{e^{-\varepsilon\left(y-\frac{1}{2\varepsilon}\right)^2}}\ \underbrace{e^{\frac{1}{4\varepsilon}}}}$$

From this we see that the integrand is bounded by an integrable function for all derivatives orders α, β

To conclude for (ii), simply note that from Theorem 1, we can move the derivatives inside the integral, we get

$$\partial_t u = \int_{-\infty}^{\infty} \partial_t \Phi(x-y,t)\, g(y)\, dy$$

$$u_{xx} = \int_{-\infty}^{\infty} \partial_{xx} \Phi(x-y,t)\, g(y)\, y$$

substituting this into heat equation gives

$$\partial_t u - D\partial_{xx} u = \int_{-\infty}^{\infty} \left[\partial_t \Phi - D\partial_{xx}\Phi\right] g(y)\, dy = 0$$

Heat equation → Derivation from Physical laws

So far → Solution on finite domains (various geometries) through separation of variables

→ Fourier series

→ Maximum principle (weak)

→ Dimensionless formulation + steady state and transient solutions

→ invariants + fundamental solution

→ Existence for Cauchy problem (+ proof)

for $|g(x)| \leq C e^{ax^2}$

Today : → Proof of Existence for Cauchy problem (part iii)

→ Duhamel formula

→ Uniqueness result for Cauchy problem

→ Random Walk

→

Laplace → Harmonic functions (including mean value property)

→ invariants (including rotations)

+ derivation of Fundamental Solution

→ Poisson's formula

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→ invariants (including rotations)

+ derivation of Fundamental Solution

→ Poisson's formula

Theorem (existence of solution to Cauchy problem)

Assume that there exist positive numbers a, c such that

$$|g(x)| \le c e^{ax^2} \quad \text{for all } x \in \mathbb{R}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi D t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4Dt}} g(y)\, dy$$

let $T < \frac{1}{4aD}$

(i) there exist positive numbers C, A s.t

$$|u(x,t)| \le C e^{Ax^2} \quad \text{for all } (x,t) \in \mathbb{R} \times (0,T]$$

(ii) $u \in C^\infty(\mathbb{R} \times (0,T])$ and in the strip $\mathbb{R} \times (0,T]$ $u_t - D u_{xx} = 0$

(iii) let $(x,t) \to (x_0, 0^+)$ If g is continuous at x_0 then

$$u(x,t) \longrightarrow g(x_0)$$

Proof of (iii)

We want to show that $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t $\underbrace{|x - x_0|, t < \delta}$

$$\underbrace{|u(x,t) - g(x_0)| < \varepsilon}$$

g continuous $\Rightarrow \forall \varepsilon/2 \ \exists \delta$ s.t $|y - x_0| < \delta$

$$|g(y) - g(x_0)| < \varepsilon/2$$

$$u(x,t) - g(x_0) = \int_{\underbrace{|y-x_0|<\delta}} \Phi(x-y, t)\,[\underbrace{g(y) - g(x_0)}_{\varepsilon/2}]\,dy$$

$$+ \int_{|y-x_0|>\delta} \Phi(x-y,t)\,[g(y)-g(x_0)]\,dy$$

$$\leq \frac{\varepsilon}{2} + \int_{|y-x_0|>\delta} \Phi(x-y,t)\,[g(y)-g(x_0)]\,dy$$

For the second term, we use

$$|\, g(y) - g(x_0) \,| \leq Ce^{ay^2} + Ce^{ax_0^2}$$

$$u(x,t) - g(x_0) \leq \frac{\varepsilon}{2} + \int_{|y-x_0|>\delta} \Phi(x-y,t)\left(Ce^{ay^2} + Ce^{ax_0^2}\right) dy$$

$$\leq \frac{\varepsilon}{2} + Ce^{ax_0^2} \int_{|y-x_0|>\delta} \Phi(x-y,t)\,dy$$

$$+\, C \int_{|y-x_0|>\delta} \Phi(x-y,t)\, e^{ay^2}\, dy$$

We focus on the blue term, the third term can be bounded in a similar way.

$$y' = \frac{x_0-y}{\sqrt{4Dt}}$$

$$C e^{ax_0^2} \int_{|y-x_0|>\delta} \underbrace{\frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{(x_0-y)^2}{4Dt}\right) dy}$$

$$|y-x_0|>\delta$$

$$\leq C e^{ax_0^2} \underbrace{\int_{|y'|>\frac{\delta}{\sqrt{4Dt}}} e^{-y^2}\, dy}$$

$$|\sqrt{4Dt}\, y'| > \delta$$

$$|y'| > \frac{\delta}{\sqrt{4Dt}}$$

We now take the $\lim_{t \to 0^+}$ of $u(x_0, t) - g(x_0)$

$$\lim_{t \to 0^+} |u(x_0, t) - g(x_0)| \leq \frac{\varepsilon}{2} + \lim_{t \to 0^+} C e^{a x_0^2} \underbrace{\int_{|y'| > \frac{\delta}{\sqrt{4 \gamma t}}} e^{-y^2} \, dy}$$

$$\lim_{t \to 0^+} |u(x_0, t) - g(x_0)| \leq \varepsilon$$

Duhamel principle

We consider a general problem of the form

$$\begin{cases} u_t - D u_{xx} = \underline{f(x,t)} & \text{in } \mathbb{R} \times (0,T] \\ u(x,0) = \underline{g(x)} & \text{in } \mathbb{R} \end{cases} \qquad (*)$$

To compute the solution, we first consider the problem with homogeneous Cauchy Data

$$\begin{cases} \underline{u_t - D u_{xx}} = f(x,t) \\ u(x,0) = 0 \leftarrow \end{cases} \qquad (*) \leftarrow$$

To solve $(*)$, we will rely on the following steps

1) Construct a family of solutions of Cauchy problems with variable initial time s and Cauchy data $f(x,s)$

2) Integrate the resulting solutions with respect to s over $(0,t)$

Together these steps are known as Duhamel's method

$$\begin{cases} w_t - D w_{xx} = 0 & x \in \mathbb{R} \quad t > \underline{s} \\ w(x,s) = \underbrace{f(x,s)} \end{cases} \qquad (*)$$

here s is viewed as a parameter

recall $\Phi(x,t)$ is a solution to the heat equation

with $\Phi(x,0) = \delta(x)$

$$\overbrace{\Phi(x,t-s)} = v(x,t)$$

$$\begin{cases} v_t - D v_{xx} = 0 \\ v(x,\underbrace{s}) = \underbrace{\delta(x)} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

$$\overbrace{\int_{-\infty}^{\infty} f(x)\,\delta(x)\, dx} = \underbrace{f(0)}$$

$$\int_{-\infty}^{\infty} f(y)\,\delta(x-y)\, dy = f(x)$$

if we consider the Cauchy problem

$$\begin{cases} w_t - D w_{xx} = 0 \\ w(x,s) = f(x,s) \end{cases}$$

(this is exactly the same setting as the theorem that we just proved)

the solution to this problem is given

$$w(x,t;s) = \int \Phi(y-x, t-s)\, f(y,s)\, dy$$

From Duhamel's method we can then define our solution to the red problem as

$$v(x,t) = \int_0^t w(x,t;s)\, ds$$

$$v(x,t) = \int_0^t \int_{\mathbb{R}} \Phi(y-x, t-s)\, f(y,s)\, dy$$

to check that $v(x,t)$ is a valid solution to $(*)$

$$v(x,0) = 0$$

$$\partial_t v - D\,\partial_{xx} v = 0$$

$$D\,\partial_{xx} v = \int_0^t \int_{\mathbb{R}} \partial_{xx} \Phi(y-x, t-s)\, f(y,s)\, dy$$

$$\partial_t v = \frac{\partial}{\partial t} \int_0^t \int_{\mathbb{R}} \Phi(y-x, t-s)\, f(y,s)\, dy$$

$$+ \int_0^t \int_{\mathbb{R}} \partial_t \Phi(y-x, t-s)\, f(y,s)\, dy$$

$$= \int_{\mathbb{R}} \Phi(y-x, 0)\, f(y,t)\, dy \qquad \text{use } \Phi(y-x,0) = \delta(y-x,0)$$

$$+ \int_0^t \int_{\mathbb{R}} \partial_t \Phi(y-x, t-s)\, f(y,s)\, dy \,.$$

$$= \overbrace{\int_{\mathbb{R}} \delta(y-x)\, f(y,t)\, dy}^{f(x,t)}$$

$$+ \int_0^t \int_{\mathbb{R}} \partial_t \Phi(y-x, t-s)\, f(y,s)\, dy$$

Grouping the ∂_t contribution with the laplacian contribution, we get

$$\partial_t v - D\, \partial_{xx} v = f(x,t) + \int_0^t \int_{\mathbb{R}} \partial_t \Phi(y-x, t-s)\, f(y,s)\, dy$$

$$- D \int_0^t \int_{\mathbb{R}} \partial_{xx} \Phi(y-x, t-s)\, f(y,s)\, dy$$

$$= f(x,t) + \int_0^t \int_{\mathbb{R}} \underbrace{(\partial_t - D\partial_{xx}) \Phi(y-x,t-s) f(y,s)\, dy}_{=0}$$

To conclude, as a solution for $(*)$ we simply combine the solution to the non homogeneous heat equation with homogeneous Cauchy data with the solution to the homogeneous heat equation with non homogeneous Cauchy data.

$$u(x,t) = \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s)\, f(y,s)\, dy$$

$$+ \int_{\mathbb{R}} \Phi(x-y,t)\, g(y)\, dy$$

at $t \to 0^+$ we recover $\int_{\mathbb{R}} \Phi(x-y,t)\, g(y)\, dy = g(x)$

$$\partial_t u - D\,\partial_{xx} u = f(x,t) + 0$$