MATH-UA 9263 - Partial Differential Equations Recitation 4: Laplace equation and Harmonic functions

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January 2022

Question 1 *Show that the Laplacian in spherical coordinates reads as*

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left\{\frac{1}{(\sin\psi)^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \psi^2} + \cot\psi\frac{\partial}{\partial \psi}\right\}$$

Question 2 (Salsa) *Show that if a function u is harmonic on a domain Ω, then the derivatives of all orders are harmonic on Ω. [Hint: you can assume that u is C^∞]*

Question 3 (Salsa) *Let B_R be the unit disk centered at $(0,0)$. Use the method of separation of variables to solve the problem*

$$\begin{cases} \Delta u = f & \text{in } B_R \\ u = 1 & \text{on } \partial B_R \end{cases}$$

Find an explicit formula when $f(x,y) = y$. [Hint: Use polar coordinates, expand $f = f(r,\cdot)$ in sine Fourier series in $[0,2\pi]$ and derive a set of ODEs for the Fourier coefficients of $u(r,\cdot)$]

Question 4 *Let u be harmonic in $\mathbb{R}^3$ and such that*

$$\int_{\mathbb{R}^3} |u(x)|^2\, dx < \infty$$

Show that $u \equiv 0$. [Hint: Write the mean value formula in a ball $B_R(0)$. Use the Schwartz inequality and let $R \to +\infty$]

Question 5 *Show that for any rotation matrix $\boldsymbol{M}$, we have, if we let $v(x) = u(\boldsymbol{M}x)$, we have*

$$\text{Tr}\left[\boldsymbol{M}^T D^2 u(\boldsymbol{M}x)\boldsymbol{M}\right] = \text{Tr}\left[D^2 u(\boldsymbol{M}x)\right]$$

Question 6 (Strauss) *An analytic function is a function that is expressible as a power series in the complex variable z. I.e.*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Show that $f(z)$ satisfies the Cauchy Riemann equations for $z = x + iy$,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and therefore, that it satisfies Laplace's equation.

Question 7 *We want to solve Laplace's equation, $\Delta\phi = 0$ within a cylindrical volume of radius a and height L. We assume that the boundary conditions that are imposed at the bounding surface are of the form*

$$\begin{cases} \phi(r,\theta,0) = 0 \\ \phi(a,\theta,z) = 0 \\ \phi(r,\theta,L) = \Phi(r,\theta) \end{cases}$$

Where $\Phi(r,\theta)$ is a given function of (r,θ). Using separation of variables, as well as the expression of the Laplacian in cylindrical coordinates, find a solution to this problem. [Hint: Use the fact that the equation $\frac{d^2R}{dp^2} + \frac{1}{p}\frac{dR}{dp} + \left(1 - \frac{m^2}{p^2}\right)R = 0$ (known as Bessel's equation) as a standard solution given by the Bessel function

$$J_m(p) = \frac{1}{\pi}\int_0^{\pi} \cos(p\sin\theta - m\theta)\, d\theta$$

]

Question 8 *Consider Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$ with the boundary conditions*

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(L,y) = g(y), \quad \frac{\partial u}{\partial y}(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,H) = f(x)$$

(a) *What is the solvability condition and its physical interpretation?*

(b) *Show that $u(x,y) = A(x^2 - y^2)$ is a solution if $f(x)$ and $g(y)$ are constants (under the condition of part (a))*

(c) *Under the conditions of part (a), solve the general case [non constant $f(x)$ and $g(y)$]. [Hint: Use part (b) and the fact that $f(x) = f_{\rm av} + [f(x) - f_{\rm av}]$, where $f_{\rm av} = \int_0^L f(x)\, dx$]*

Question 9 (Poisson's formula) *We consider Laplace's equation on a disk in* $\mathbb{R}^2$*. That is, let* $\Omega = \{(x,y) \in \mathbb{R}^2 \, : \, x^2 + y^2 < a^2\}$*. Consider the Boundary value problem*

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ u = h(\theta) & (x,y) \in \partial\Omega \end{cases}$$

Show that the solution to this problem is given by Poisson's formula

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 + r^2 - 2ar\cos(\theta - \phi)} d\phi$$

[Keep in mind that we don't want a solution that blows up as $r \to 0^+$*]*

Question 10 *Determine the general form of the solution of the following boundary value problem*

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = a^2 \dfrac{\partial u}{\partial t} + f(x) \\ u(0,t) = a \\ u(L,t) = b \\ u(x,0) = g(x) \end{cases}$$

[Hint: one technique used in solving equations of the type given above consists in introducing a new function of the form $w(x,t) + \psi(x)$ *where* $w(x,t)$ *satisfies a homogeneous PDE and* $\psi(x)$ *is determined from the solution of an ODE]*

Question 11 (Salsa) *We say that a function* $u \in C^2(\Omega)$*,* $\Omega \subseteq \mathbb{R}^n$ *is subharmonic (resp. superharmonic) in* Ω *if* $\Delta u \geq 0$ *(resp.* $\Delta u \leq 0$*) in* Ω*. Show that*

1. *If* u *is subharmonic, then, for every* $B_R(\boldsymbol{x}) \subset\subset \Omega$*,*

$$u(\boldsymbol{x}) \leq \frac{n}{\omega_n R^n} \int_{B_R(\boldsymbol{x}) u(\boldsymbol{y}) \, d\boldsymbol{y}}$$

and

$$u(\boldsymbol{x}) \leq \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(\boldsymbol{x})} u(\boldsymbol{y}) \, d\boldsymbol{y}$$

If u *is superharmonic, the reverse inequalities hold.*

2. *if* $u \in C(\overline{\Omega})$ *is subharmonic, (resp. superharmonic), the maximum (resp. minimum) of* u *is attained on* $\partial\Omega$

3. *Let* u *be subharmonic in* Ω *and* $F : \mathbb{R} \to \mathbb{R}$*, smooth. Under which conditions on* F *is* $F \circ u$ *harmonic?*